Row operations, invertible matrices, row reduced echelon form
Last time: Let $A$ be a matrix and let $B$ be the matrix where we take $A$ and double the first row:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}.$$ 

Then $A$ and $B$ have the same kernel. The image of $B$ is the set of vectors $[2x, y]$ for $[x, y]$ in the image of $A$. 
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Then $A$ and $B$ have the same kernel. The image of $B$ is the set of vectors $\left[ \begin{array}{c} 2x \\ y \end{array} \right]$ for $\left[ \begin{array}{c} x \\ y \end{array} \right]$ in the image of $B$.

We can write this more clearly in terms of the matrix $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. So $B = DA$.

Our statements are

$$\text{Ker}(DA) = \text{Ker}(A) \quad \text{and} \quad \text{Im}(DA) = D\text{Im}(A).$$
The key property of $D$ is that it is *invertible*. This means that there is a matrix $C$, called the *inverse* of $D$, with

$$CD = \text{Id} \text{ and } DC = \text{Id}.$$  

We write $C = D^{-1}$.

In this case where $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, the inverse matrix is $\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$.

We’ll see soon that only square matrices can have inverses.
Can you show?

**Theorem** If $A$ and $B$ are invertible, then $AB$ is invertible.

**Theorem** A matrix $U$ can only have one inverse. In other words, if $UV = UW = \text{Id}$ and $VU = WU = \text{Id}$, then $V = W$. 
Theorem If $A$ and $B$ are invertible, then $AB$ is invertible.

Proof: We claim that $B^{-1}A^{-1}$ is the inverse of $AB$. Indeed:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIdA^{-1} = AA^{-1} = Id$$ and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IdB = B^{-1}B = Id.$$  \[\square\]

The inverse of “put on your socks, put on your shoes” is “take off your shoes, take off your socks”.
**Theorem** A matrix $U$ can only have one inverse. In other words, if $UV = UW = \text{Id}$ and $VU = WU = \text{Id}$, then $V = W$.

**Proof:** Consider $VUW$. We have $VUW = (VU)W = \text{Id}W = W$ but also $VUW = V(UW) = V\text{Id} = V$ so $W = V$. □
Now, back to our main point:

**Theorem** Let $E$ be an invertible $m \times m$ matrix and let $A$ be any $m \times n$ matrix. Then $\text{Ker}(EA) = \text{Ker}(A)$ and $\text{Im}(EA) = E\text{Im}(A)$.

In other words, $EA\vec{x} = \vec{0}$ if and only if $A\vec{x} = \vec{0}$. For $\vec{y}$ in $\mathbb{R}^m$, there is an $\vec{x}$ with $A\vec{x} = \vec{y}$ if and only if there is a $\vec{w}$ with $EA\vec{w} = E\vec{y}$. 
Theorem Let $E$ be an invertible $m \times m$ matrix and let $A$ be any $m \times n$ matrix. Then $\text{Ker}(EA) = \text{Ker}(A)$ and $\text{Im}(EA) = E\text{Im}(A)$. In other words, $EA\vec{x} = \vec{0}$ if and only if $A\vec{x} = \vec{0}$. For $\vec{y}$ in $\mathbb{R}^m$, there is an $\vec{x}$ with $A\vec{x} = \vec{y}$ if and only if there is a $\vec{w}$ with $EA\vec{w} = E\vec{y}$.

Proof: If $A\vec{x} = \vec{0}$ then $EA\vec{x} = E\vec{0} = \vec{0}$. On the other hand, if $EA\vec{x} = \vec{0}$, then $E^{-1}EA\vec{x} = \text{Id}A\vec{x} = A\vec{x}$ so $A\vec{x} = E^{-1}\vec{0} = \vec{0}$.

If $A\vec{x} = \vec{y}$ then $EA\vec{x} = E\vec{y}$. Conversely, if $EA\vec{x} = E\vec{y}$ then $E^{-1}EA\vec{x} = E^{-1}E\vec{y}$, meaning that $A\vec{x} = \vec{y}$. □
**Theorem** Let $E$ be an invertible $m \times m$ matrix and let $A$ be any $m \times n$ matrix. Then $\text{Ker}(EA) = \text{Ker}(A)$ and $\text{Im}(EA) = E\text{Im}(A)$.

Take

$$E = \begin{bmatrix} 1 & 1 & \cdots & c \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 \\ \end{bmatrix} \quad c \neq 0.$$  

We get the row operation of multiplying a single row by a nonzero scalar.
This is one of three row operations:

1. Multiply a row by a nonzero scalar.
2. Switch two rows.
3. Add a multiple of one row to another row.
All of the row operations are multiplication by invertible matrices:

1. Multiply a row by a nonzero scalar.

\[
\begin{bmatrix}
1 \\
\vdots \\
c \\
\vdots \\
1
\end{bmatrix}
\]

2. Switch two rows.

\[
\begin{bmatrix}
0 & 1 \\
\vdots & \vdots \\
1 & 1 \\
1 & 0
\end{bmatrix}
\]

3. Add a multiple of one row to another row.

\[
\begin{bmatrix}
1 \\
\vdots \\
1 \\
a \vdots \\
1
\end{bmatrix}
\]
Thus, if we get from one matrix $A$ to another matrix $B$ by row operations, then $\text{Ker}(A) = \text{Ker}(B)$ and the images of $A$ and $B$ are related in a natural way. There will be an invertible matrix $U$ with $B = UA$. 
Thus, if we get from one matrix $A$ to another matrix $B$ by row operations, then $\text{Ker}(A) = \text{Ker}(B)$ and the images of $A$ and $B$ are related in a natural way. There will be an invertible matrix $U$ with $B = UA$.

This raises the natural question: How nice can we make a matrix, using row operations? The answer is row reduced echelon form.
Row reduced echelon form

• Either row is either all 0’s, or else its first nonzero entry is a 1. This 1 is called a *pivot*.

• In a column which contains a pivot, called a *pivot column*, all the other entries are 0.

• The nonzero rows are at the top of the matrix; they are ordered so that the pivots go from left to right as we go down the rows.

```
\begin{bmatrix}
1 & 3 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 2 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
```
Putting a matrix into row reduced echelon form

To make a matrix into row reduced echelon form (rref), we work from left to right. Look at the leftmost column which is not yet a pivot column, and which has a nonzero in a non-pivot row.

• Rescale that entry to be 1.
• Subtract appropriate multiples of the row with the 1 from other rows to make the other entries of that column be 0.
• Switch rows, if needed, to put that row immediately below the already existing pivot rows.

\[
\begin{bmatrix}
3 & 6 & 3 & 12 \\
1 & 2 & 4 & 13 \\
2 & 4 & 4 & 14
\end{bmatrix}
\mapsto
\begin{bmatrix}
1 & 2 & 1 & 4 \\
1 & 2 & 4 & 13 \\
2 & 4 & 4 & 14
\end{bmatrix}
\mapsto
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & 0 & 3 & 9 \\
0 & 0 & 2 & 6
\end{bmatrix}
\mapsto
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & 0 & 1 & 3 \\
0 & 0 & 2 & 6
\end{bmatrix}
\mapsto
\begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Vocabulary related to rref

• The initial leading 1’s are called *pivots*. The columns that contain them are called *pivot columns*; the corresponding variables in our system of equations are called *pivot variables*.

• The columns/variables which are not pivot columns/variables are called *free columns/variables*.

• The number of pivots is called the *rank*.