Theorems about invertible matrices
Remember last time: A matrix is in row reduced echelon form if:

- Either row is either all 0’s, or else its first nonzero entry is a 1. This 1 is called a pivot.

- In a column which contains a pivot, called a pivot column, all the other entries are 0.

- The nonzero rows are at the top of the matrix; they are ordered so that the pivots go from left to right as we go down the rows.
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- In a column which contains a pivot, called a *pivot column*, all the other entries are 0.
- The nonzero rows are at the top of the matrix; they are ordered so that the pivots go from left to right as we go down the rows.

**Theorem:** For any matrix $A$, there is an invertible matrix $U$ such that $UA$ is in row reduced echelon form.
**Theorem:** Let $A$ be an $m \times n$ matrix. The following are equivalent:

1. Every column of $\text{rref}(A)$ is a pivot column.
2. There is an $n \times m$ matrix $B$ with $BA = \text{Id}_n$.
3. $\text{Ker}(A) = \{\vec{0}\}$.
4. For all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, if $A\vec{x} = A\vec{y}$ then $\vec{x} = \vec{y}$. (Vocabulary: $A$ is injective.)
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Note: If these conditions hold, then $m \geq n$. 
Proofs: (1) \implies (2): If every column is a pivot column, the row reduced form of $A$ must be

$$R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So there is some invertible $U$ with $UA = R$.

Now, $QR = \text{Id}_n$ where

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

So $(QU^{-1})(UR) = \text{Id}_n$. Take $B = QU^{-1}$. 
(2) $\implies$ (3): Suppose that $A\vec{x} = \vec{0}$. Then $BA\vec{x} = B\vec{0} = \vec{0}$ so we deduce that $\vec{x} = \vec{0}$.

(3) $\implies$ (4): Suppose that $A\vec{x} = A\vec{y}$. Then $A(\vec{x} - \vec{y}) = 0$, so $\vec{x} - \vec{y} = \vec{0}$ and we deduce that $\vec{x} = \vec{y}$.
NOT(1) ⇒ NOT(4): Let \( R \) be the row reduced form of \( A \), and let the \( k \)-th column be a free column. Let the pivot columns be \( p_1, p_2, \ldots, p_r \).

\[
R = \begin{bmatrix}
1 & 0 & R_{1k} & * & 0 \\
0 & 1 & R_{2k} & * & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Then \( Re_k = R \sum_j R_{jk} e_{p_j} \). In the example,

\[
\begin{bmatrix}
1 & 0 & R_{1k} & * & 0 \\
0 & 1 & R_{2k} & * & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix}
1 & 0 & R_{1k} & * & 0 \\
0 & 1 & R_{2k} & * & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix} R_{13} \\ R_{23} \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Then \( URe_k = UR \sum_j R_{jk} e_{p_j} \), where \( A = UR \). \(\Box\)
Theorem: Let $A$ be an $m \times n$ matrix. The following are equivalent:

1. Every row of $\text{rref}(A)$ is a pivot row.
2. There is an $n \times m$ matrix $C$ with $AC = \text{Id}_m$.
3. Image($A$) = $\mathbb{R}^m$. (Vocabulary: $A$ is surjective).
Theorem: Let $A$ be an $m \times n$ matrix. The following are equivalent:

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3. $\text{Image}(A) = \mathbb{R}^m$. (Vocabulary: $A$ is surjective).

Note, if these conditions happen, then $m \leq n$. 

Proofs: (1) \(\implies\) (2): Let \(R\) be the row reduced form, and let \(A = UR\). So \(R\) looks like

\[
R = \begin{bmatrix}
1 & * & 0 & * \\
0 & 0 & 1 & *
\end{bmatrix}.
\]

Then \(RQ = \text{Id}\), where

\[
Q = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

And then \((UR)(QU^{-1}) = \text{Id}\) as well.
(2) $\implies$ (3): Suppose that $AC = \text{Id}_m$. Let $\vec{y}$ be in $\mathbb{R}^m$. Then we claim that $\vec{x} = C\vec{y}$ is a solution to $A\vec{x} = \vec{y}$. Indeed, $AC\vec{y} = \text{Id}\vec{y} = \vec{y}$.

\textbf{NOT}(1) $\implies$ \textbf{NOT}(3). Let $R$ be the row reduced form, and let $A = UR$. Suppose that the bottom row of $R$ is not a pivot row. Then $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the image of $R$. So $U \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the image of $A$. 
Theorem: Let $A$ be an $m \times n$ matrix. The following are equivalent:

1. Every column of $\text{rref}(A)$ is a pivot column.
2. There is an $n \times m$ matrix $B$ with $BA = \text{Id}_n$.
3. $\text{Ker}(A) = \{ \vec{0} \}$.
4. $A$ is injective.

If these conditions hold, then $m \geq n$.

Theorem: Let $A$ be an $m \times n$ matrix. The following are equivalent:

1. Every row of $\text{rref}(A)$ is a pivot row.
2. There is an $n \times m$ matrix $C$ with $AC = \text{Id}_m$.
3. $A$ is surjective,

If these conditions hold, then $m \leq n$. 
Corollary: If $A$ has an inverse, then $m = n$.

Corollary: If $A$ is injective and surjective, then $m = n$. 
Theorem Let \( m = n \) and let \( A \) be an \( m \times n \) matrix. Then the following are equivalent:

1. Every column of \( \text{rref}(A) \) is a pivot column.
2. There is an \( n \times m \) matrix \( B \) with \( BA = \text{Id}_n \).
3. \( \text{Ker}(A) = \{ \vec{0} \} \).
4. \( A \) is injective.
5. Every row of \( \text{rref}(A) \) is a pivot row.
6. There is an \( n \times m \) matrix \( C \) with \( AC = \text{Id}_m \).
7. \( A \) is surjective,
**Theorem** Let $m = n$ and let $A$ be an $m \times n$ matrix. Then the following are equivalent:

1. Every column of $\text{rref}(A)$ is a pivot column.
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5. Every row of $\text{rref}(A)$ is a pivot row.
6. There is an $n \times m$ matrix $C$ with $AC = \text{Id}_m$.
7. $A$ is surjective,

**Proof:** We already did the equivalence of $(1) - (4)$, and of $(5) - (7)$. For square matrices, $(1)$ and $(5)$ are obviously equivalent.