V a vector space over a field F

A multilinear form is a function whose inputs are k vectors from V and whose output is a scalar \( F \) which is linear in each variable.

\[
m(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)
\]

\[
m(x \vec{v}_2, \ldots, \vec{v}_k) = m(x, \vec{v}_2, \ldots, \vec{v}_k) + m(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)
\]

\[
m(c \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = cm(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)
\]

These conditions hold for any input position, not just the first.

If \( e_1, e_2, \ldots, e_n \) is a basis of V, then \( m \) is uniquely determined by the \( n \times k \) values

\[
m(e_{[i_1]}, e_{[i_2]}, \ldots, e_{[i_k]})
\]

So the vector space of \( k \)-linear forms on \( V \) has dimension \( n \times k \).
Suppose we have an $a$-linear form $\alpha$ on $V$ and a $b$-linear form $\beta$. We can make an $(a+b)$-linear form by tensoring them together.

$$\left(\alpha \otimes \beta\right)(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{a+b}) =$$

$$\alpha(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_a) \beta(\vec{v}_{a+1}, \vec{v}_{a+2}, \ldots, \vec{v}_{a+b})$$

Let $V$ have basis $e_1, e_2$.
Let $a=b=1$, so $\alpha$ and $\beta$ are in the dual space $V^*$.
Let $\alpha = e_1^* + 2e_2^*$
$\beta = 3e_1^* + 4e_2^*$.

$$\left(\alpha \otimes \beta\right)(\begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} r \\ s \end{bmatrix}) =$$

$$\alpha(\begin{bmatrix} p \\ q \end{bmatrix}) \beta(\begin{bmatrix} r \\ s \end{bmatrix}) = (p + 2q)(3r + 4s)$$

$$\alpha(\begin{bmatrix} p \\ q \end{bmatrix}) = p \alpha(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + q \alpha(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = p \cdot 1 + q \cdot 2$$
\[ \alpha, \beta : V \rightarrow F \]

Bilinear form on V.

\[ \alpha \otimes \beta \]

An alternating form is uniquely determined by its value on \((e_{(1)}, e_{(2)}, \ldots, e_{(k)})\) for \(1 \leq i_1 < i_2 < \ldots < i_k \leq n\).

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

So the vector space of alternating forms has dimension \((n \text{ choose } k)\).

This is alternating (also known as skew-symmetric or anti-symmetric).

In other words:

\[ (\alpha \wedge \beta)(u, v) = \alpha(u) \beta(v) - \beta(u) \alpha(v) \]

In particular, if \(k = n\), the space of alternating forms is one dimensional, and gives the determinant.

\[ \alpha \wedge \alpha \wedge \ldots \wedge \alpha = \sum_{\sigma \in S_k} \text{sign}(\sigma) \alpha_{c_{\sigma(1)}} \otimes \alpha_{c_{\sigma(2)}} \otimes \ldots \otimes \alpha_{c_{\sigma(k)}}. \]

\[ \text{sign}(\sigma) = (-1)^{\# \text{ of pairs } (i, j) \text{ with } i < j \text{ and } \sigma(i) > \sigma(j)}. \]

\[ \alpha \wedge \beta \wedge \gamma = \alpha \otimes \beta \otimes \gamma - \alpha \otimes \gamma \otimes \beta + \gamma \otimes \beta \otimes \alpha - \gamma \otimes \alpha \otimes \beta + \beta \otimes \alpha \otimes \gamma - \beta \otimes \gamma \otimes \alpha - \gamma \otimes \beta \otimes \alpha + \gamma \otimes \alpha \otimes \beta. \]
Let $V$ be an $n$-dimensional vector space. Then there is a 1-dimensional space of alternating $n$-linear forms on $V$.

If we choose a basis $e_1, e_2, ..., e_n$ for $V$, then such an alternating form takes $(v_1, v_2, ..., v_n)$ to
\[
\det \left[ \begin{array}{c}
  v_1 \\
  v_2 \\
  \vdots \\
  v_n 
\end{array} \right] .
\]

All alternating forms are a multiple of this one.

Given $T : V \to V$, we get a scalar called $\det(T)$. If $\omega$ is any nonzero alternating $n$-linear form, then
\[
\omega(Tv_1, Tv_2, ..., Tv_n) = \det(T) \omega(v_1, v_2, ..., v_n).
\]

Basic properties of $\det$:
\[
\det(AB) = \det(A) \det(B).
\]

* Adding a multiple of one row/column to another doesn't change determinant.

* Rescaling a single row/column by a scalar $c$ multiplies determinant by $c$.

* Switching two rows/columns changes the sign of the determinant.

For $T : V \to V$, we have $\det(T) = 0$ if and only if $T$ is not invertible. Equivalently, $T$ has a kernel. Equivalently, $T$ does not have image $= V$.

\[
\det(v_1, v_2, ..., v_n) = 0
\]
if and only if $v_1, v_2, ..., v_n$ are NOT a basis. Equivalently, $v_1, v_2, ..., v_n$ are linearly dependent. Equivalently, they don't span $V$. 

\[
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{bmatrix}
\]

\[
\begin{bmatrix}
  e & f \\
  h & i
\end{bmatrix}
\]

\[
\begin{bmatrix}
  d & f \\
  g & i
\end{bmatrix}
\]

\[
\begin{bmatrix}
  d & e \\
  g & h
\end{bmatrix}
\]
Let $e_1, e_2, \ldots, e_n$ be the standard basis of $V$.

Then

$$(e_1^* \wedge e_2^* \wedge \ldots \wedge e_n^*)(v_1, v_2, \ldots, v_n) \text{ is } \det(v_1, v_2, \ldots, v_n).$$

Concretely, expanding this gives the formula for determinant as a sum of permutations.

Abstractly, we know that there is only a one dimensional space of alternating forms on $V$, so $(e_1^* \wedge e_2^* \wedge \ldots \wedge e_n^*)$ must be proportional to determinant, and it isn’t hard to check what the scalar is.

\[
(e_1^* \wedge e_2^* \wedge \ldots \wedge e_n^*)(v_1, v_2, \ldots, v_n) = \\
((e_1^*) \wedge (e_2^* \wedge \ldots \wedge e_n^*))(v_1, v_2, \ldots, v_n) = \\
e_1^*(v_1) (e_2^* \wedge \ldots \wedge e_n^*)(v_2, v_3, \ldots, v_n) - \\
e_1^*(v_2) (e_2^* \wedge \ldots \wedge e_n^*)(v_1, v_3, \ldots, v_n) + \\
e_1^*(v_3) (e_2^* \wedge \ldots \wedge e_n^*)(v_1, v_2, \ldots, v_n) - \\
\ldots + \\
e_1^*(v_n) (e_2^* \wedge \ldots \wedge e_n^*)(v_1, v_2, \ldots, v_{n-1})
\]

This is the row expansion of $\det(v_1, v_2, \ldots, v_n)$ along row 1.
Cayley - Hamilton:

Let $T : V \rightarrow V$ be a linear transformation and let $\chi_T$ be its characteristic polynomial.

$$\chi_T(x) = \det (xI - T).$$

Then $\chi_T(T) = 0$.

$$T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$xI - T = \begin{bmatrix} x-1 & -2 \\ -3 & x-4 \end{bmatrix}$$

$$\chi_T(x) = \det \begin{bmatrix} x-1 & -2 \\ -3 & x-4 \end{bmatrix} = (x-1)(x-4) + (-2)(-3) = x^2 - 5x + 10 - 6 = x^2 - 5x - 2$$

$$T^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$5T = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$

$$2T = 2 \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$T^2 - 5T + 2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \chi_A(x) = \det \begin{bmatrix} x-1 & -1 \\ -1 & x \end{bmatrix} =
\]

\[
x(x-1) - 1 = x^2 - x - 1
\]

So \(A^2 - A - \text{Id} = 0\).

So \(A^{n+2} - A^{n+1} - A^n = 0\) or

\[
A^{n+2} = A^{n+1} + A^n
\]

\[
A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = A^2 + A
\]

\[
A^4 = A^3 + A^2 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}
\]

The powers of this matrix are matrices of Fibonacci numbers!

\[
\begin{bmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{bmatrix}
\]
Special cases of Cayley-Hamilton:

If $A$ is diagonalizable, may as well assume diagonal:

$$A = \begin{bmatrix}
\lambda_1 \\
& \lambda_2 \\
& & \ddots \\
& & & \lambda_n
\end{bmatrix}$$

$$f_A(x) = \text{det}(xI_n - A) = \text{det} \begin{bmatrix}
x - \lambda_1 \\
& x - \lambda_2 \\
& & \ddots \\
& & & x - \lambda_n
\end{bmatrix} = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

$$X_A(A) = \begin{bmatrix}
(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \\
(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \\
(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \\
0 & 0 & \ddots & \ddots & \ddots
\end{bmatrix}$$
Special case of Cayley-Hamilton:
Suppose there is a vector \( v \) such that \( v, T v, T^2 v, \ldots, T^{(n-1)} v \) form a basis of \( V \).

In that case,

\[
T^{n}(v) = c_{(n-1)} T^{(n-1)} v + \ldots + c_1 T v + c_0 v
\]

for some scalars \( c_0, c_1, \ldots, c_{(n-1)} \).

In this basis, the matrix of \( T \) is

\[
\begin{bmatrix}
0 & \sigma & \sigma & \cdots & c_0 \\
0 & \sigma & \sigma & \cdots & c_1 \\
0 & 0 & \sigma & \cdots & c_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sigma \end{bmatrix}
\]

The characteristic polynomial is

\[
x^n - c_{(n-1)} x^{(n-1)} - \ldots - c_1 x - c_0.
\]

Want to check that

\[
T^{n} - c_{(n-1)} T^{(n-1)} - \ldots - c_1 T - c_0 \text{Id} = 0.
\]

Just need to check that it sends each basis vector to 0.

\[
T^{n} (T^{j} v) - c_{(n-1)} T^{(n-1)} (T^{j} v) - \ldots - c_1 T (T^{j} v) - c_0 (T^{j} v) =
T^{(n+j)} v - c_{(n-1)} T^{(n-1+j)} v - \ldots - c_0 T^{j} v =
(T^{(n+j)} - c_{(n-1)} T^{(n-1+j)}) v - \ldots - c_0 T^{j} v =
T^{j} (T^{n} - c_{(n-1)} T^{n-1} - \ldots - c_0 \text{Id}) v =
T^{j} \ast 0 = 0.
\]
General case, is that we show that any linear transformation can be put in the form

$$\begin{bmatrix}
T_i & & T_{Z_i} & \\
 & & & \\
 & & T_k & \\
\end{bmatrix}$$

where each $T_k$ has a cyclic vector.

Just like in the diagonalizable case,

$$\chi_T(x) = \text{product of } \chi_{[T_k]}(x).$$

By the cyclic vector case,

$$\chi_{[T_k]}(T_k) = 0.$$

We use this to show that

$$\chi_T(T) = 0.$$

Block upper triangularization $\rightarrow$ factorization of characteristic polynomial

Primary decomposition theorem says that (factorization of $\chi_T$ or minimal polynomial) $\rightarrow$ (block diagonalization)
Primary decomposition theorem.

Let $V$ be finite dimensional vector space. Let $T : V \rightarrow V$ be a linear transformation. Let $g(x)$ be a polynomial with $g(T) = 0$.

Suppose $g(x)$ factors as $f_1(x) f_2(x) \ldots f_k(x)$ with $\gcd(f_i(x), f_j(x)) = 1$ for $i$ and $j$ relatively prime.

Put $W_i = \text{Ker}(f_i)$. Then we have

* $V$ is the direct sum of the $W_i$
* $T$ maps $W_i$ to $W_i$
* $T$ restricted to $W_i$ obeys $f_i$.

In coordinates, we can block diagonalize $T$. Each block gives $T$ restricted to $W_i$. 
Quotient spaces:

Let $V$ be a vector space, $W$ a subspace, $V/W$ is the quotient space.

If $v_1, v_2, \ldots, v_n$ is a basis for $V$ such that $v_1, v_2, \ldots, v_k$ is a basis for $W$, then $v_{k+1}, v_{k+2}, \ldots, v_n$ is a basis for $V/W$.

In particular, $\dim(V/W) = \dim(V) - \dim(W)$.

We always have a surjective linear map $V \longrightarrow V/W$ whose kernel is $W$.

If we have another linear map $A: V \longrightarrow X$ with $A(W) = 0$, then it factors through $V/W$.

In particular, suppose that we have $B: V \longrightarrow V$ with $B$ sending $W$ to $W$. Then, in bases $v_1, v_2, \ldots, v_n$ as before, $B$ looks like

$$B = \begin{bmatrix} P & Q \\ 0 & R \end{bmatrix}$$

where $P$ is the matrix of $W \longrightarrow W$ and $R$ is the matrix of $V/W \longrightarrow V/W$. 