Problem Set Ten: Due Thursday, April 7 at 11:59 PM

See course website for homework policies.

Reading Read 8.1-8.4.

Textbook problems Please solve 8.2.1, 8.2.2, 8.2.12, 8.4.4, 8.4.8.

Problem 1. In this problem, we will prove the following result: Let $A$ be a square matrix and suppose that the characteristic polynomial $\chi_A(x)$ factors into linear factors $\chi_A(x) = \prod (x - \lambda_i)^{n_i}$. Then there is a basis in which $A$ is upper triangular. (Of course, if all the eigenvalues are already distinct, we know that $A$ is diagonalizable.)

1. Let $V$ be an $m$-dimensional vector space and let $C : V \to V$ be a linear transformation with $C^m = 0$. Show that $V$ has a basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ such that $C(\vec{v}_i) \in \text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1})$. Conclude that, in this basis, $C$ is upper triangular with 0’s on the diagonal.

2. Let $V$ be an $m$-dimensional vector space, let $\lambda$ be a scalar and let $B : V \to V$ be a linear transformation with $\chi_B(x) = (x - \lambda)^m$. Show that there is a basis for $V$ in which $B$ is upper triangular with $\lambda$’s on the diagonal.

3. Let $V$ be an $m$-dimensional vector space, let $A : V \to V$ be a linear transformation and suppose that the minimal polynomial $\chi_A(x)$ factors into linear factors $\chi_A(x) = \prod (x - \lambda_i)^{n_i}$. Show that there is a basis for $V$ where $A$ is upper triangular with the $\lambda_i$ on the diagonal.

Problem 2. Let $F$ be a field and let $f(x) = x^n + f_{n-1}x^{n-1} + \cdots + f_1x + f_0$ be an irreducible polynomial with coefficients in $F$.

1. Let $V$ be an $n$-dimensional vector space and let $A : V \to V$ be a linear transformation with $\chi_A(x) = f(x)$. Let $\vec{v}$ be any nonzero vector in $V$. Show that $\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}$ is a basis of $V$.

2. Let $A$ and $V$ be as in the previous part. Write the matrix of $A$ in the basis $\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}$.

Problem 3. Let $V$ be the vector space of continuous functions on $[-\pi, \pi]$. Define an inner product on $V$ by

$$\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$ 

1. Show that the following list of functions is orthonormal: $\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(nx)$ for $n \geq 1$, and $\frac{1}{\sqrt{\pi}} \cos(nx)$ for $n \geq 1$.

2. Let $f(x) = x$. Find the function in $\text{Span}(\sin x, \sin(2x), \sin(3x))$ which is closest to the function $f(x)$. As an incentive, here is a plot of $y = x$ and of $y = a_1 \sin(x) + a_2 \sin(2x) + a_3 \sin(3x)$ for optimally chosen $a_1, a_2, a_3$. 

![Plot of functions](image.png)