Problem 1. Recall that the cross product of two vectors in \( \mathbb{R}^3 \) is defined by 
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2y_3 - y_2x_3 \\ y_2x_1 - x_2y_1 \\ x_1y_2 - y_1x_2 \end{bmatrix}.
\]
For any vector \( \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \) in \( \mathbb{R}^3 \), define \( B_\vec{c}(\vec{x}, \vec{y}) := \vec{c} \cdot (\vec{x} \times \vec{y}) \).

(1) Show that, for any vector \( \vec{c} \in \mathbb{R}^3 \), the function \( B_\vec{c}(\ , \ ) \) is an alternating bilinear form.
(2) Let \( B(\ , \ ) \) be any alternating bilinear form on \( \mathbb{R}^3 \). Show that there is a unique vector \( \vec{c} \in \mathbb{R}^3 \) such that \( B(\ , \ ) \) is \( B_\vec{c}(\ , \ ) \).

Solution: Part 1: We must check:

- Additivity: \( \vec{c} \cdot ((\vec{x}_1 + \vec{x}_2) \times \vec{y}) = \vec{c} \cdot (\vec{x}_1 \times \vec{y} + \vec{x}_2 \times \vec{y}) = \vec{c} \cdot (\vec{x}_1 \times \vec{y}) + \vec{c} \cdot (\vec{x}_2 \times \vec{y}) \) and \( \vec{c} \cdot (\vec{x} \times (\vec{y}_1 + \vec{y}_2)) = \vec{c} \cdot (\vec{x} \times \vec{y}_1 + \vec{x} \times \vec{y}_2) = \vec{c} \cdot (\vec{x} \times \vec{y}_1) + \vec{c} \cdot (\vec{x} \times \vec{y}_2) \).
- Scalar multiplication: \( \vec{c} \cdot ((a\vec{x}) \times \vec{y}) = a\vec{c} \cdot (\vec{x} \times \vec{y}) = \vec{c} \cdot (\vec{x} \times (a\vec{y})) \).
- Alternation: \( \vec{c} \cdot (\vec{x} \times \vec{x}) = \vec{c} \cdot \vec{0} = 0 \).

Part 2: We first write down a general alternating bilinear form. Let \( e_1, e_2, e_3 \) be the standard basis of \( \mathbb{R}^3 \). Then \( B(e_1, e_1) = B(e_2, e_2) = B(e_3, e_3) = 0 \) and define \( B(e_1, e_2) = -B(e_2, e_1) = p, B(e_1, e_3) = -B(e_3, e_1) = r, B(e_2, e_3) = -B(e_3, e_2) = s \). Then we have

\[
B(x_1e_1 + x_2e_2 + x_3e_3, y_1e_1 + y_2e_2 + y_3e_3) = \sum_{i,j=1}^{3} x_iy_j B(e_i, e_j)
= p(x_1y_2 - x_2y_1) + q(x_1y_3 - x_3y_1) + r(x_2y_3 - x_3y_2).
\]
This is \( \begin{bmatrix} r & q \\ -p & -q \end{bmatrix} \cdot (\vec{x} \times \vec{y}) \), so we take \( \vec{c} = \begin{bmatrix} r & q \\ -p & -q \end{bmatrix} \) (and no other \( \vec{c} \) works).

Problem 2. Let \( V \) be an \( n \) dimensional vector space over a field \( F \). Let \( e_1, e_2, \ldots, e_n \) be one basis for \( V \) and let \( f_1, f_2, \ldots, f_n \) be another basis. Let \( S \) be the matrix defined by \( f_j = \sum_i S_{ij} e_i \).

(1) Let \( T : V \rightarrow V \) be a linear map and define the matrices \( X \) and \( Y \) by \( T(e_j) = \sum_i X_{ij} e_i \) and \( T(f_j) = \sum_i Y_{ij} f_i \). Give a formula for \( Y \) in terms of \( X \) and \( S \).
(2) Show that \( \det X = \det Y \).
(3) Let \( B : V \times V \rightarrow F \) be a bilinear form and define the matrices \( P \) and \( Q \) by \( B(e_i, e_j) = P_{ij} \) and \( B(f_i, f_j) = Q_{ij} \). Give a formula for \( Q \) in terms of \( P \) and \( S \).
(4) Show that there is a nonzero element \( s \in F \) with \( \det P = s^2 \det Q \).

Solution Part 1: There are slicker ways to do this computation, but I’ll write out the brute force solution. We have

\[
T(f_j) = \sum_i Y_{ij} f_i = \sum_i Y_{ij} \left( \sum \limits_h S_{hi} e_h \right) = \sum \limits_{\ell,i} S_{\ell i} Y_{ij} e_\ell.
\]
But also

\[
T(f_j) = T \left( \sum \limits_k S_{kj} e_k \right) = \sum \limits_k S_{kj} T(e_k) = \sum \limits_k S_{kj} \left( \sum \limits_\ell X_{\ell k} e_\ell \right) = \sum \limits_{k,\ell} X_{\ell k} S_{kj} e_\ell.
\]
Since the \( e \)'s are a basis, we can set their coefficients equal to get

\[
\sum_i S_{\ell i} Y_{ij} = \sum_k X_{\ell k} S_{kj}.
\]
Written as a matrix equation, this says that $SY = XS$, so $Y = S^{-1}XS$.

**Part 2:** Taking determinants, we get $\det(Y) = \det(S)^{-1} \det(X) \det(S) = \det(X)$.

**Part 3:** We compute

$$Q_{ij} = B(f_i, f_j) = B \left( \sum_h S_{hi} e_h, \sum_k S_{kj} e_k \right) = \sum_{h,k} S_{hi} S_{kj} B(e_h, e_k) = \sum_{h,k} S_{hi} S_{kj} P_{hk}.$$ 

We can write this as a matrix equation: $Q = S^TPS$.

**Part 4:** We have $\det(Q) = \det(S^T) \det(P) \det(S) = \det(S)^2 \det(P)$ where $s = \det(S)$.

**Problem 3.** Let $V$ be an $n$-dimensional vector space over a field $F$ and let $B : V \times V \to F$ be an alternating bilinear form. In this problem, we will show that there is some integer $r$ such that there is a basis $\vec{x}_1, \vec{y}_1, \vec{x}_2, \vec{y}_2, \ldots, \vec{x}_r, \vec{y}_r, \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{n-2r}$ of $V$ such that $B(\vec{x}_i, \vec{y}_i) = -B(\vec{y}_i, \vec{x}_i) = 1$ and all other pairings between the basis vectors are 0. This proof is by induction on $n$.

1. Do the base cases $n = 1$ and $n = 2$.
2. Explain why we are done if $B(\vec{v}, \vec{w}) = 0$ for all vectors $\vec{v}$ and $\vec{w}$ in $V$.

From now on, assume that $n > 2$ and that $B(\vec{v}, \vec{w})$ is not always 0. Choose two vectors $\vec{x}, \vec{y}$ with $B(\vec{x}, \vec{y}) = 1$. Set $V' = \{\vec{v} : B(\vec{x}, \vec{v}) = B(\vec{y}, \vec{v}) = 0\}$.

3. Show that $V = \text{Span}(\vec{x}, \vec{y}) \oplus V'$.
4. By induction, $V'$ has a basis $\vec{x}_1, \vec{y}_1, \vec{x}_2, \vec{y}_2, \ldots, \vec{x}_r, \vec{y}_r, \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{n-2r}$ as required. Explain how to finish the proof from here.
5. We conclude with an example. Consider the alternating bilinear form

$$B((u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4)) = \sum_{1 \leq i < j \leq 4} (u_i v_j - u_j v_i)$$

on $\mathbb{R}^4$. Find a basis $\vec{x}_1, \vec{y}_1, \vec{x}_2, \vec{y}_2$ as above.

**Solution Part 1:** When $n = 1$, the form $B$ must be 0, since there is only one basis vector $e_1$ and we have $B(e_1, e_1) = 0$. So we take $r = 0$ and $\vec{z}_1 = e_1$. Now, take $n = 2$. Let $e_1, e_2$ be a basis for $V$. If $B(e_1, e_2) = 0$, then $B$ is 0, so we can take $r = 0$ with $\vec{z}_1 = e_1$ and $\vec{z}_2 = e_2$. If $B(e_1, e_2) = b \neq 0$, then we can take $r = 1$ with $\vec{x}_1 = e_1$ and $\vec{y}_1 = e_2/b$.

**Part 2:** We just take $r = 0$ and take $\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_n$ to be any basis of $V$.

**Part 3:** We need to check that $\text{Span}(\vec{x}, \vec{y}) \cap V' = \{0\}$ and $V = \text{Span}(\vec{x}, \vec{y}) + V'$. For the first, consider a vector $a\vec{x} + b\vec{y}$. This vector will be in $V'$ if and only if $B(\vec{x}, a\vec{x} + b\vec{y}) = B(\vec{y}, a\vec{x} + b\vec{y}) = 0$. We expand $B(\vec{x}, a\vec{x} + b\vec{y}) = bB(\vec{x}, \vec{y}) = 0$ and $B(\vec{y}, a\vec{x} + b\vec{y}) = aB(\vec{y}, \vec{x}) = -a$. So $-a = b = 0$ and the vector is 0.

Now, we must show that any vector $\vec{v} \in V$ is of the form $(a\vec{x} + b\vec{y}) + \vec{w}$ for $\vec{w} \in V'$. Indeed, take $\vec{w} = \vec{v} - B(\vec{v}, \vec{y})\vec{x} - B(\vec{x}, \vec{v})\vec{y}$. We need to check that $\vec{w}$ is in $V'$. We have

$$B(\vec{x}, \vec{w}) = B(\vec{x}, \vec{v} - B(\vec{v}, \vec{y})\vec{x} - B(\vec{x}, \vec{v})\vec{y}) = B(\vec{x}, \vec{v}) - B(\vec{v}, \vec{y})B(\vec{x}, \vec{y}) - B(\vec{x}, \vec{v}) = 0.$$

$$B(\vec{y}, \vec{w}) = B(\vec{y}, \vec{v} - B(\vec{v}, \vec{y})\vec{x} - B(\vec{x}, \vec{v})\vec{y}) = B(\vec{y}, \vec{v}) - B(\vec{v}, \vec{y})B(\vec{y}, \vec{x}) + B(\vec{y}, \vec{v})B(\vec{y}, \vec{x}) = 0.$$

**Part 4:** We simply take the basis $\vec{x}_1, \vec{y}_1, \vec{x}_2, \vec{y}_2, \ldots, \vec{x}_r, \vec{y}_r, \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{n-2r}$. We constructed $V'$ such that $B(\vec{x}, \vec{v}) = B(\vec{y}, \vec{v}) = 0$ for all $\vec{v}$ in $V'$, which means that $\vec{x}$ and $\vec{y}$ have the correct pairing with all the other basis vectors in this list. They also pair to 1 with each other and all the other vectors, inductively, have the correct pairing, so the result holds.
Part 5: We carry out the algorithm implied in the inductive procedure. Put $\vec{x}_1 = (1, 0, 0, 0)$ and $\vec{y}_1 = (0, 1, 0, 0)$. Let $V' = \{ \vec{v} : B(\vec{x}_1, \vec{v}) = B(\vec{y}_1, \vec{v}) = 0 \}$. We compute explicitly that $V'$ is the set of $(v_1, v_2, v_3, v_4)$ such that

\[ v_2 + v_3 + v_4 = -v_1 + v_3 + v_4 = 0. \]

We see that a basis of $V'$ is $(1, -1, 1, 0), (1, -1, 0, 1)$. Then $B((1, -1, 1, 0), (1, -1, 0, 1)) = (-1) + 1 - (-1) + (-1) - 1 - (-1) + 1 = 1$. So we can take $\vec{x}_2 = (1, -1, 1, 0)$ and $\vec{y}_2 = (1, -1, 0, 1)$. 