SOLUTION SET NINE

6.6.2 By assumption, every vector in $V$ can be written in the form $\vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_k$ for $\vec{w}_i \in W_i$, so what we want to show is that there is only one such expression. There are many ways to do this, but here is a particularly nice way: Let $W_1 \oplus W_2 \oplus \cdots \oplus W_k$ be the abstract direct sum of the $W_i$. So $\dim(W_1 \oplus W_2 \oplus \cdots \oplus W_k) = \sum \dim W_i$. We have a map $W_1 \oplus W_2 \oplus \cdots \oplus W_k \to V$ by $(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \mapsto \vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_k$. Our assumption that $V = W_1 + W_2 + \cdots + W_k$ means that this map is surjective. But both vector spaces have the same dimension, so this shows it is also injective.

6.6.4 False. We’ll take our field to be $\mathbb{R}$. The map $[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]$ is a projection onto $\mathbb{R}[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$ and the map $[\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}]$ is a projection onto $\mathbb{R}[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}]$. But $[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}] + [\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}] = [\begin{smallmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}]$ is not a projection. Indeed, we check that $[\begin{smallmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}]^2 = [\begin{smallmatrix} 5/2 & 1 \\ 1 & 1/2 \end{smallmatrix}] \neq [\begin{smallmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}]$.

6.6.6 True. Choosing a basis where our operator is diagonal, we can assume $E$ is of the form

$$
\begin{bmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & 0 \\
& & & 0 \\
\end{bmatrix}
$$

Then $E^2 = E$ as desired.

6.7.1 Suppose first that $ETE = TE$ and let $\vec{y} \in \text{Im}(E)$, meaning that $\vec{y} = E\vec{x}$ for some $\vec{x}$. Then $T\vec{y} = T(TE\vec{x}) = E(TTE\vec{x}) = E(TE\vec{x})$, so $T\vec{y} \in \text{Im}(E)$ as desired.

Conversely, suppose that $T$ maps $\text{Im}(E)$ to itself. For any vector $\vec{x}$, we then have $TE\vec{x} \in \text{Im}(E)$. But $E$ maps every vector in $\text{Im}(E)$ to itself, so we deduce that $E(TE\vec{x}) = TE\vec{x}$, as desired.

Now, suppose that $ET = TE$. This implies that $ETE = TE^2 = TE$, so we conclude that $T$ maps $\text{Im}(E)$ to itself as desired. We must check that $T$ also maps $\text{Ker}(E)$ to itself. Indeed, let $E\vec{z} = 0$. Then $E(T\vec{z}) = TE\vec{z} = T\vec{z} = \vec{0}$, showing that $T\vec{z}$ is in $\text{Ker}(E)$ as desired.

Finally, suppose that $T$ maps $\text{Im}(E)$ and $\text{Ker}(E)$ to themselves. Since $V = \text{Im}(E) \oplus \text{Ker}(E)$, every vector in $V$ can be written as $\vec{y} + \vec{z}$ for $\vec{y} \in \text{Im}(E)$ and $\vec{z} \in \text{Ker}(E)$. By the definition of $\text{Im}(E)$, we rewrite this as $\vec{v} = E\vec{x} + \vec{z}$. We must check that $ET(E\vec{x} + \vec{z}) = TE(E\vec{x} + \vec{z})$. On the left hand side, we have $ET(E\vec{x} + \vec{z}) = ETE\vec{x} + ET\vec{z}$. By our assumption on $T$, the vector $TE\vec{x}$ is in $\text{Im}(E)$, so $ETE\vec{x} = TE\vec{x}$. Also by our assumption on $T$, the vector $T\vec{z}$ is in $\text{Ker}(E)$, so $ET\vec{z} = \vec{0}$. We conclude that the left hand side is $TE\vec{x}$. On the right hand side, we have $TE(E\vec{x} + \vec{z}) = TE^2\vec{x} + \vec{0} = TE\vec{x}$. So both sides are $TE\vec{x}$ and we are done.

6.7.2 (a) We check that $[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}] [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}] = [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}] = 2 [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$.

(b) Any space complementary to $\mathbb{R}[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$ must be of the form $\mathbb{R}[\begin{smallmatrix} x \\ 1 \end{smallmatrix}]$ for some $x$. But then $[\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}] [\begin{smallmatrix} x \\ 1 \end{smallmatrix}] = [\begin{smallmatrix} 2x + 1 \\ 1 \end{smallmatrix}]$. If $\mathbb{R}[\begin{smallmatrix} x \\ 1 \end{smallmatrix}]$ were invariant, then the vector $[\begin{smallmatrix} 2x + 1 \\ 1 \end{smallmatrix}]$ would have to be in $\mathbb{R}[\begin{smallmatrix} x \\ 1 \end{smallmatrix}]$. But $[\begin{smallmatrix} 2x + 1 \\ 1 \end{smallmatrix}] = 1 \neq 0$, so there is no $x$ for which $[\begin{smallmatrix} 2x + 1 \\ 1 \end{smallmatrix}]$ is a multiple of $[\begin{smallmatrix} x \\ 1 \end{smallmatrix}]$.

6.8.1 The characteristic polynomial of $T$ is $x^3 - 2x^2 + x - 2 = (x-2)(x^2+1)$. We take $p_1(x) = x - 2$ and $p_2(x) = x^2 + 1$. We compute bases for $p_1(T)$ and $p_2(T)$.

We have

$$
p_1(T) = T - 2\text{Id}_3 = \begin{bmatrix} 4 & -3 & -2 \\ -1 & 0 & -5 \\ 1 & 10 & -5 \end{bmatrix} = \mathbb{R} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
$$

$$
p_2(T) = T^2 + \text{Id}_3 = \begin{bmatrix} 5 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & 10 & 0 \end{bmatrix} = \mathbb{R} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
$$

6.8.9 One example is

$$
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Both of these matrices have minimal polynomial $x^2$. 