Problem 1 Let \( G \) be a connected graph with equally many vertices and edges. Show that \( G \) has exactly one cycle.

Problem 2 Let \( G \) be a directed graph on a finite vertex set \( V \).
(a) Suppose that every vertex of \( G \) has out-degree 1. Show that \( G \) has a directed cycle.
(b) Suppose that \( v \in V \) is a vertex of out-degree 0 and every vertex other than \( v \) has out-degree 1. Show that the following are equivalent:
   (i) \( G \), considered as an undirected graph, is connected
   (ii) \( G \), considered as an undirected graph, is a tree
   (iii) \( G \), considered as an undirected graph, has no cycles
   (iv) \( G \), considered as a directed graph, has no directed cycles

Problem 3 Let \( T \) be a tree all of whose vertices have degree either 1 or 3. Such a tree is called \textit{trivalent} and often occur in evolutionary biology, describing how various species have branched apart from each other.

(a) If \( T \) has \( n \) leaves, show that it has \( n - 2 \) vertices of degree 3.
(b) Let \( T \) be a trivalent tree with \( n \geq 4 \). Show that there is some internal vertex which is adjacent to two leaves.

Such a vertex is sometimes called a \textit{cherry}, and many algorithms for phylogenetic reconstruction begin by trying to find the cherries.

Problem 4 Let \( T \) be a tree.
(a) Show that it is possible to color the vertices of \( T \) black and white so that neighboring vertices have opposite colors.
   Let \( b \) and \( w \) be the numbers of black and white vertices.
   (b) If \( b \geq w \), show that \( T \) has a black leaf.
   (c) Let \( \ell \) be the number of leaves of \( T \). Show that \( |b - w| < \ell \) (unless \( T \) is a single vertex).

Problem 5 Consider trivalent trees (defined in Problem 2) whose leaves are numbered 1, 2, \ldots, \( n \). We consider two such trees \( T \) and \( T' \) to be the same if there is an isomorphism \( T \cong T' \) preserving the labels of the leaves. Below, we depict the three trees for \( n = 4 \).

(a) How many such trees are there for \( n = 5, 6 \) and 7? (Don’t write them out!)
(b) Conjecture a formula for the number of such trees with \( n \) leaves.
(c) Prove your guess.

Problem 6 Consider a \((2n + 1) \times (2n + 1)\) checkerboard. Place \(2n^2 + 2n\) dominos on the checker board, leaving one corner uncovered. Show that it is possible to slide the dominos in order to move the hole to any position whose \( x \) and \( y \) coordinates have the same parity as the initial corner. (These positions are marked with red dots in the image.)