Problem Set 6 – due October 29nd

Please see the course website for homework policy.

Problem 1 Let $G$ be a graph and let $L(G)$ be the Laplacian matrix of $G$. Let $\lambda_2(G)$ be the second smallest eigenvalue of $L(G)$. (The smallest eigenvalue is 0, corresponding to the all 1 vector.)

Hint (ROT13): Guvf ceboyrz vf nobhg Gur Enyrvtu dhbgvrag.

(a) Let $G'$ be a graph formed by adding an extra edge to $G$. Show that $\lambda_2(G') \geq \lambda_2(G)$.

(b) I am no longer sure this is true, though I can’t break it. Please let me know if you find a proof.

Let $v$ be a vertex of $G$, with neighbors $u_1, u_2, \ldots, u_k$. Choose $i$ with $1 \leq i \leq k$ and make a new graph $G'$ as follows: replace $v$ by two vertices $x$ and $y$, where $x$ is joined to $u_1, u_2, \ldots, u_i$ and $y$ is joined to $u_{i+1}, u_{i+2}, \ldots, u_k$. Show that $\lambda_2(G') \leq \lambda_2(G)$.

Problem 2 Let $H$ be a bipartite graph and let $A(H)$ be its adjacency matrix. Show that, if $\lambda$ is an eigenvalue of $H$, then $-\lambda$ is also an eigenvalue of $H$.

Problem 3 Let $G$ be a $d$-regular graph on $n$ vertices. Let $A(G)$ be the adjacency matrix of $G$, with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) Show that $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = dn$.

(b) Show that $\max(|\lambda_2|, |\lambda_n|) \geq \sqrt{\frac{dn-d^2}{n-1}}$.

(c) Fix $d$ and fix $\epsilon > 0$. Show that, for $n$ sufficiently large, $\max(|\lambda_2|, |\lambda_n|) > \sqrt{d} - \epsilon$.

Problem 4 (a) Let $G$ be the $n$-cycle and let $L(G)$ be the Laplacian matrix of $G$. Show that the eigenvalues of $L(G)$ are $2 - 2\cos(2\pi j/n)$, for $0 \leq j < n$.

(b) Let $H$ be the path of length $n$ and let $L(H)$ be the Laplacian matrix of $H$. Let $G$ be the $2n$-cycle with Laplacian matrix $L(G)$. Show that every eigenvalue of $L(H)$ is also an eigenvalue of $L(G)$. More specifically, show that the eigenvalues of $H$ are $2 - 2\cos(2\pi j/(2n))$, for $0 \leq j < n$.

(c) Let $S$ be the tree with $n+1$ vertices, one of which borders all the others. Find the eigenvalues of $L(S)$.