Problem Set 7 – due November 19th

You have two weeks for this problem set, since there will be no class on November 7 and since November 5 will be a wrap up of the material on spectral graph theory. Problems 1 and 2 are designed to use only cleverness without theory.

Please see the course website for homework policy.

Problem 1 Consider a \((2n+1) \times (2n+1)\) checker board, with the corners colored black. Suppose that we remove any black square from the board, living \(4n^2 + 4n\) squares behind. Show that the remaining squares can be tiled with dominos. (A direct proof is probably easier than appealing to Hall’s marriage theorem.)

Problem 2 Let \(G\) be a graph. By definition, a perfect matching of \(G\) is a collection \(M\) of edges of \(G\) so that every vertex of \(G\) lies on exactly one edge of \(M\). Let \((v_1, v_2, \ldots, v_{2k})\) be a cycle of \(G\) of even length. If \(M\) is a perfect matching of \(G\) which contains the edges \((v_1, v_2), (v_3, v_4), \ldots, (v_{2k-1}, v_{2k})\), then define the twist of \(M\) along \((v_1, v_2, \ldots, v_{2k})\) to be the perfect matching which deletes the edges \((v_1, v_2), (v_3, v_4), \ldots, (v_{2k-1}, v_{2k})\) from \(M\) and replaces them by the edges \((v_2, v_3), (v_4, v_5), \ldots, (v_{2k-2}, v_{2k-1}), (v_{2k}, v_1)\).

(a) If \(M\) and \(M'\) are two perfect matchings of \(G\), show that we can change \(M\) to \(M'\) by a sequence of twists along various cycles of \(G\).

An induced cycle of \(G\) is a cycle \((v_1, v_2, \ldots, v_m)\) so that \(G\) has no edges among the vertices \(v_i\) other than the \(m\) edges of the cycle.

(b) Prove or disprove: If \(M\) and \(M'\) are two perfect matchings of \(G\), then we can change \(M\) to \(M'\) by a sequence of twists along various induced cycles of \(G\).

(c) Prove or disprove: If \(G\) is bipartite and \(M\) and \(M'\) are two perfect matchings of \(G\), then we can change \(M\) to \(M'\) by a sequence of twists along various induced cycles of \(G\).

Let \(G\) be a graph drawn in the plane without crossing itself. We’ll say that a cycle of \(G\) bounds a face if there are no edges of \(G\) inside the part of \(\mathbb{R}^2\) enclosed by that cycle.

(d) (Harder) Prove or disprove: Let \(G\) be a connected bipartite planar graph and let \(M\) and \(M'\) be two perfect matchings. Then we can change \(M\) to \(M'\) by a sequence of twists along cycles which bound faces.

Problem 3 We deal a deck of cards into 13 piles of 4. Show that it is possible to pick up one card from each pile and have precisely one card of each rank: One ace, one deuce and so forth, up to one king.

Problem 4 Let \(A\) be a square \(n \times n\) matrix. I’ll say that \(N\) is a magic square if there is some constant \(N\) such that every row and every column\(^1\) of \(A\) sums to \(N\). A permutation matrix is a \((0,1)\) matrix where every row and every column contains exactly one 1; so a permutation matrix is a magic square with \(N=1\).

(a) Show that, if \(A\) is a magic square with nonnegative integer entries, then \(A\) is a sum of \(N\) permutation matrices. (Hint in ROT13: Svefg gel gb fubj gung gurer vf n fvatyr crezhgungyba zngevk P fhpun gung A – P fgvyy unf abaartngvir vagtrre ragevrf.)

(b) Show that, if \(A\) is a magic square with nonnegative real entries, then \(A\) is a positive linear combination of permutation matrices.

(c) (Harder) Let \(A\) be an \(n \times n\) magic square with nonnegative real entries. Show that there are \((n-1)^2 + 1\) permutation matrices, \(P_1, P_2, \ldots, P_{(n-1)^2 + 1}\) so that \(A = \sum c_i P_i\) with the \(c_i\) nonnegative real numbers.

\(^1\)We don’t impose this condition on the diagonals.