Let $R$ be a ring and let $M$ be an $R$-module.

**Definition:** A chain of submodules of $M$ is a sequence $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$. We call $\ell$ the **length** of the chain.

**Definition:** A **composition series** is a chain of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ such that each quotient module $M_i/M_{i-1}$ is simple. We recall that the zero module is not considered simple, so $M_i \neq M_{i+1}$ in a composition series.

**Problem 10.1.** Suppose that there is a positive integer $L$ such that, for any chain $0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$, we have $\ell \leq L$. Show that $M$ has a composition series. (Hint: Consider a chain of maximal length.)

**Definition:** We say that $M$ has **finite length** if $M$ has a composition series.

**Problem 10.2.** Let $M$ be an $R$-module which is a finite set. Show that $M$ has finite length.

**Problem 10.3.** Let $k$ be a field which is contained in the center of $R$. (Meaning that, for all $a \in k$ and $r \in R$, we have $ar = ra$.) Suppose that $M$ is finite dimensional as a $k$-vector space. Show that $M$ has finite length.

The following nonstandard definition will be convenient:

**Definition:** A **quasi-composition series** is a chain of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$ such that each quotient module $M_i/M_{i-1}$ is either simple or $0$.

**Problem 10.4.** Show that, if $M$ has a quasi-composition series, then $M$ has a composition series.

**Problem 10.5.** Let $\alpha : A \hookrightarrow B$ be an injective $R$-module homomorphism, and let $0 = B_0 \subset B_1 \subset \cdots \subset B_0 = B$ be a composition series. Show that $\alpha^{-1}(B_0) \subseteq \alpha^{-1}(B_1) \subseteq \cdots \subseteq \alpha^{-1}(B_0)$ is a quasi-composition series.

**Problem 10.6.** Let $\beta : B \twoheadrightarrow C$ be a surjective $R$-module homomorphism, and let $0 = B_0 \subset B_1 \subset \cdots \subset B_0 = B$ be a composition series. Show that $\beta(B_0) \subseteq \beta(B_1) \subseteq \cdots \subseteq \beta(B_0)$ is a quasi-composition series.

This, the property of having a composition series passes to submodules and to quotient modules.