Worksheet 13: Unique Factorization Domains (UFDs)

Throughout this worksheet, let \( R \) be an integral domain.

**Definition:** Let \( r \) be an element of \( R \). We say that \( r \) is **composite** if \( r \) is nonzero and \( r \) can be written as a product of two non-units. We say that \( r \) is **irreducible** if it is neither composite, nor 0, nor a unit.

Thus every element of \( R \) is described by precisely one of the adjectives “zero”, “unit”, “composite”, “irreducible”.

**Definition:** Let \( p \in R \). We say that \( p \) is **prime** if \( pR \) is a prime ideal and \( p \neq 0 \).

**Problem 13.1.** Let \( p \) be a non-zero, non-unit. Show that \( p \) is prime if and only if, whenever \( p|ab \), either \( p|a \) or \( p|b \).

**Problem 13.2.** Show that prime elements are irreducible.

**Problem 13.3.** Let \( k \) be a field and let \( k[t^2, t^3] \) be the subring of \( k[t] \) generated by \( t^2 \) and \( t^3 \).

1. Check that \( t^2 \) and \( t^3 \) are not prime in \( k[t^2, t^3] \).
2. Show that \( t^2 \) and \( t^3 \) are prime in \( k[t^2, t^3] \).

**Problem 13.4.** Consider the subring \( \mathbb{Z}[\sqrt{-5}] \) of \( \mathbb{C} \).

1. Show that 2, 3 and \( 1 \pm \sqrt{-5} \) are irreducible in \( \mathbb{Z}[\sqrt{-5}] \). Hint: Use the complex absolute value.
2. Show that 2, 3 and \( 1 \pm \sqrt{-5} \) are not prime in \( \mathbb{Z}[\sqrt{-5}] \). Hint: \( 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}) \).

We want to say that factorizations into prime elements are unique, but factorizations into irreducible elements need not be. In order to do this, we need some vocabulary.

**Definition:** We define two elements, \( p \) and \( q \), of \( R \) to be **associate** if there is a unit \( u \) such that \( p = qu \). We define two factorizations \( p_1 p_2 \cdots p_m \) and \( q_1 q_2 \cdots q_n \) to be **equivalent** if \( m = n \) and there is a permutation \( \sigma \) in \( S_n \) such that \( p_j \) is associate to \( q_{\sigma(j)} \).

**Problem 13.5.** Show that any non-zero, non-unit element of \( R \) has at most one factorization into prime elements, up to equivalence.

**Problem 13.6.** Give examples, in the rings \( k[t^2, t^3] \) and \( \mathbb{Z}[\sqrt{-5}] \), of elements with multiple, nonequivalent, factorizations into irreducible elements.

**Definition:** We’ll make the following nonstandard definition: We’ll say that \( R \) has factorizations if every non-zero, non-unit \( R \) can be written in at least one way as a product of irreducibles.

**Problem 13.7.** Let \( R \) have factorizations. Show that the following conditions are equivalent:

(a) All irreducible elements are prime.
(b) Factorizations into irreducibles are unique, up to equivalence.
(c) Every nonzero, nonunit, element has a factorization into prime elements.

**Definition:** An integral domain which has factorizations and in which the equivalent conditions in Problem 13.7 hold, is called a **unique factorization domain**, also known as a UFD.

**Problem 13.8.** Let \( R \) be a Noetherian integral domain.

1. Let \( r_1, r_2, r_3 \ldots \) be a sequence of elements of \( R \) such that \( r_{j+1} \) divides \( r_j \) for all \( j \). Show that, for \( j \) sufficiently large, \( r_j \) and \( r_{j+1} \) are associates.
2. Show that \( R \) has factorizations.

1 Morally, we should consider the product of the empty set to be 1, so 1 has a factorization into a set of irreducibles, namely the empty set. But trying to get this right would be a notational pain, so we’ll just refuse to consider factorizations of units.