Most people find the proof of the Smith normal form theorem for Euclidean domains more intuitive than the case of a general PID. When I went to write them out, they actually came out very similar.

**Problem 18.1. (Proof of Smith normal form for Euclidean integral domains)** Let \( R \) be a Euclidean integral domain with positive norm \( N(\cdot) \). Let \( X \in \text{Mat}_{m \times n}(R) \). If \( X = 0 \), the Smith normal form theorem clearly holds for \( X \), so assume otherwise. Let \( d \) be an element of smallest norm among all nonzero elements occurring as an entry in a matrix \( Y \) with \( Y \sim X \). Let \( Y \) be a matrix with \( Y \sim X \) and \( Y_{11} = d \).

1. Show that \( d \) divides \( Y_{1i} \) and \( Y_{ij} \) for all \( 2 \leq i \leq m \) and \( 2 \leq j \leq n \).
2. Show that there is a matrix \( Z \sim Y \) with \( Z_{11} = d \) and \( Z_{i1} = Z_{1j} = 0 \) for all \( 2 \leq i \leq m \) and \( 2 \leq j \leq n \).
3. Show that \( d \) divides \( Z_{ij} \) for all \( 2 \leq i \leq m \) and \( 2 \leq j \leq n \).
4. Show that \( X \) is \( \sim \)-equivalent to a matrix of the form \( \text{diag}_{\min}(d_1, d_2, \ldots, d_{\min(m,n)}) \) with \( d_1 | d_2 | \cdots | d_{\min(m,n)} \).

**Problem 18.2.** Consequence of the proof of Smith normal form for Euclidean integral domains: Define a stronger equivalence relation \( \sim_E \) where \( X \sim_E Y \) if \( Y = UXV \) where \( U \) and \( V \) products of elementary matrices.

1. Trace through your proof and check that you have shown, in a Euclidean integral domain, that every matrix is \( \sim_E \)-equivalent to a matrix of the form \( \text{diag}_{\min}(d_1, d_2, \ldots, d_{\min(m,n)}) \) with \( d_1 | d_2 | \cdots | d_{\min(m,n)} \).
2. Let \( R \) be a Euclidean integral domain. Let \( SL_n(R) \) be the group of \( n \times n \) matrices with entries in \( R \) and determinant 1. Show that \( SL_n(R) \) is generated by elementary matrices.

To do the case of a general PID, you’ll need the following old problems:

14.9 Let \( x \) and \( y \in R \). Show that there is a matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) with entries in \( R \) such that \( ad - bc = 1 \) and \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \text{GCD}(x,y) \\ 0 \end{bmatrix} \).

14.10 Let \( x \) and \( y \) be nonzero elements of \( R \). Show that there are invertible \( 2 \times 2 \) matrices \( U \) and \( V \) with \( U \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} V = \begin{bmatrix} \text{GCD}(x,y) & 0 \\ 0 & \text{LCM}(x,y) \end{bmatrix} \).

Here \( \text{LCM}(x,y) = \frac{xy}{\text{GCD}(x,y)}. \)

**Problem 18.3.**
Let \( R \) be a Noetherian ring (such as a PID) and let \( \mathcal{D} \) be a nonempty subset of \( R \). Show that there is an element \( d \in \mathcal{D} \) which is “minimal with respect to division”: More precisely, show that there is an element such that if \( d' \in \mathcal{D} \) divides \( d \), then \( d \) divides \( d' \) as well.

**Problem 18.4. (Proof of Smith normal form for PID’s)** Let \( R \) be a PID and let \( X \in \text{Mat}_{m \times n}(R) \). Let \( \mathcal{D} \) be the set of all entries occurring in any matrix \( Y \) with \( Y \sim X \). Let \( d \) be as in Problem 18.3 for \( \mathcal{D} \) and let \( Y \) be a matrix with \( Y \sim X \) and \( Y_{11} = d \).

1. Show that \( d \) divides \( Y_{1i} \) and \( Y_{ij} \) for all \( 2 \leq i \leq m \) and \( 2 \leq j \leq n \).
2. Show that there is a matrix \( Z \sim Y \) with \( Z_{11} = d \) and \( Z_{i1} = Z_{1j} = 0 \) for all \( 2 \leq i \leq m \) and \( 2 \leq j \leq n \).
3. Show that \( d \) divides \( Z_{ij} \) for all \( 2 \leq i \leq m \) and \( 2 \leq j \leq n \).
4. Show that \( X \) is \( \sim \)-equivalent to a matrix of the form \( \text{diag}_{\min}(d_1, d_2, \ldots, d_{\min(m,n)}) \) with \( d_1 | d_2 | \cdots | d_{\min(m,n)} \).