**Worksheet 3: Ideals**

**Definition:** Suppose $R$ is a ring. A subset $I \subset R$ is called a left ideal provided that
- $I_1$: $(I, +)$ is a subgroup of $(R, +)$; and
- $I_2$: for all $r \in R$ we have $rI \subset I$, that is $rx \in I$ for all $x \in I$.

It is called a right ideal provided that
- $I_1$: $(I, +)$ is a subgroup of $(R, +)$; and
- $I_2$: for all $r \in R$ we have $Ir \subset I$, that is $yr \in I$ for all $y \in I$.

A subset of $R$ that is both a left and right ideal is called a two-sided ideal.

If $R$ is commutative, then “left ideal”, “right ideal” and “two-sided ideal” are the same, and we will simply write ideal.\(^1\)

**Problem 3.1.** Show that if $A$ and $B$ are ideals, then $A + B := \{a + b : a \in A, b \in B\}$ is also an ideal.

**Problem 3.2.** Fix $n \geq 2$. Let $I$ be the subset of $R = \text{Mat}_{n \times n}(\mathbb{Q})$ consisting of matrices with nonzero entries only in the first row. Is $I$ a left ideal? Is it a right ideal?

**Problem 3.3.** Suppose $R$ and $S$ are rings and $\varphi \in \text{Hom}(R, S)$. Show that $\ker(\varphi)$ is a two-sided ideal of $R$.

**Problem 3.4.** Let $R$ be a ring and let $I$ be a left ideal. Since $I$ and $R$ are abelian groups with respect to $+_R$, we can form the quotient group $R/I$. Show that $R/I$ has a natural structure as a left $R$-module.

**Problem 3.5.** Let $R$ be a ring and let $I$ be a two sided ideal. Show that $R/I$ has a natural ring structure.

\(^1\)In this course, we will not use the word “ideal” in a non-commutative ring without saying whether it is a left ideal, right ideal or two-sided ideal. If you see a source using “ideal” by itself in a non-commutative setting, it probably means “two-sided ideal”, but Prof. Speyer recommends being clearer and not using the word “ideal” by itself in this context.