**Worksheet 4: Integral Domains**

**Definition:** A commutative ring $R$ is called an integral domain if:

ID1: Whenever $xy = 0$ in $R$, we have either $x = 0$ or $y = 0$ and
ID2: The ring $R$ is not the zero ring.

Integral domains are similar to fields, but not as nice. The next problems explore the relationship.

**Problem 4.1.** Show that a field is an integral domain.

**Problem 4.2.** Show that $\mathbb{Z}$ is an integral domain but not a field.

**Problem 4.3.** Show that $k[x]$ is an integral domain but not a field, where $k$ is a field.

**Problem 4.4.** Let $R$ be a nonzero commutative ring.

1. Show that $R$ is an integral domain if and only if, for all $x \neq 0$ in $R$, the map $y \mapsto xy$ is injective.
2. Show that $R$ is a field if and only if, for all $x \neq 0$ in $R$, the map $y \mapsto xy$ is bijective.

**Problem 4.5.** Let $R$ be an integral domain and suppose that $\#(R)$ is finite. Show that $R$ is a field.

**Problem 4.6.** Let $R$ be an integral domain and let $k$ be a subring of $R$ which is a field, such that $R$ is finite dimensional as a $k$-vector space. Show that $R$ is a field.

Every integral domain $R$ embeds in a natural field, known as the field of fractions of $R$ and denoted $\text{Frac}(R)$.

**Definition:** Let $R$ be an integral domain. Define $X$ to be the set of pairs $(p, q)$ in $R^2$ with $q \neq 0$. Define an equivalence relation $\sim$ on $X$ by

$$(p_1, q_1) \sim (p_2, q_2) \text{ if and only if } p_1q_2 = p_2q_1.$$

We will denote an element of $X/\sim$ as $p/q$. We define addition and multiplication on $X/\sim$ by:

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1q_2 + p_2q_1}{q_1q_2}, \quad \frac{p_1}{q_1} * \frac{p_2}{q_2} = \frac{p_1p_2}{q_1q_2}.$$

We denote this field $\text{Frac}(R)$.

**Problem 4.7.** Verify that $\sim$ is an equivalence relation on $X$.

**Problem 4.8.** Verify that $X/\sim$ is a field under the operations $+$ and $*$ on $X/\sim$.

At this point, we can see why it is a good idea to define $\{0\}$ not to be an integral domain: If we try these definitions with $R = \{0\}$, then $X = \emptyset$, so $\text{Frac}(R)$ would be $\emptyset$ and, in particular, would not have additive or multiplicative identities.