Worksheet 7: Comaximal Ideals

We now introduce the notion of comaximal ideals. As we will see, ideals being comaximal is something like integers being relatively prime.

**Definition:** Suppose $R$ is a commutative ring. Ideals $A$, $B$ of $R$ are said to be **comaximal** provided that $A + B = R$.

**Problem 7.1.** Show that $A$ and $B$ are comaximal if and only if $1 \in A + B$.

**Problem 7.2.** If $m$ is maximal and $I$ is an ideal, show that either $m$ and $I$ are comaximal, or else $I \subseteq m$.

**Problem 7.3.** Let $R$ be a commutative ring and let $A$ and $B$ be ideals. Show that the map $R \to R/A \times R/B$, sending $r$ to the ordered pair $(r \mod A, r \mod B)$, is surjective if and only if $A$ and $B$ are comaximal.

**Definition:** Suppose $R$ is a ring. The **product** of ideals $A$ and $B$ in $R$ is the ideal, denoted $AB$, consisting of all finite sums $\sum a_i b_i$ with $(a_i, b_i) \in A \times B$. The product of any finite number of ideals is defined similarly.

**Problem 7.4.** (This one is a little tricky:) Suppose that $A$ and $B$ are comaximal ideals in a commutative ring $R$. Show that $A \cap B = AB$.

**Problem 7.5.** Suppose that $R$ is a nonzero commutative ring. Suppose $I_1$, $I_2$, $I_3$, \ldots, $I_k$ are ideals in $R$ that are pairwise comaximal. Show that the ideals $I_1$ and $I_2 I_3 \cdots I_k$ are comaximal.

We now show that comaximal is a stronger condition than relatively prime, and is the same in $\mathbb{Z}$.

**Problem 7.6.** Let $R$ be a commutative ring, let $a$ and $b$ in $R$, and suppose that $aR$ and $bR$ are comaximal. Show that any $g$ which divides both $a$ and $b$ must be a unit.

**Problem 7.7.** Show that the ideals $xk[x, y]$ and $yk[x, y]$ are **not** comaximal, although the polynomials $x$ and $y$ are relatively prime in $k[x, y]$.

**Problem 7.8.** Let $a$ and $b$ be relatively prime integers. Show that the ideals $a\mathbb{Z}$ and $b\mathbb{Z}$ are comaximal.