Worksheet 8: The Chinese Remainder Theorem

“There are certain things whose number is unknown. If we count them by threes, we have two left over; by fives, we have three left over; and by sevens, two are left over. How many things are there?” – Sunzi Suanjing (3rd century)

A lot of results today are quick citations to past worksheets! Have them ready!

**Problem 8.1.** Let $R$ be a commutative ring and let $A$ and $B$ be ideals. Describe the “obvious” map $R \rightarrow R/A \times R/B$ and show that its kernel is $A \cap B$.

**Problem 8.2.** Show that, if $R$ is a commutative ring and $A$ and $B$ are comaximal ideals, then $R/AB \cong R/A \times R/B$.

**Problem 8.3.** *(The Chinese Remainder Theorem)* Show that, if $I_1, I_2, \ldots, I_k$ are a list of pairwise comaximal ideals, then

$$R/ (I_1 I_2 \cdots I_k) \cong R/I_1 \times R/I_2 \times \cdots \times R/I_k.$$

**Problem 8.4.** Show that, if $m_1, m_2, \ldots, m_k$ are a list of pairwise relatively prime integers, then

$$\mathbb{Z}/m_1 \cdots m_k \mathbb{Z} \cong \mathbb{Z}/m_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/m_k \mathbb{Z}.$$  

**Problem 8.5.** Let $k$ be a field and $a_1, a_2, \ldots, a_r$ be distinct elements of $k$. Show that

$$k[t]/(t - a_1)(t - a_2) \cdots (t - a_r)k[t] \cong k \times \cdots \times k$$

where the right hand side has $r$ factors.