**Bilinear forms**

Suppose $k$ is a field and $V$ is a $k$-vector space.

**Definition.** A $k$-bilinear form on $V$ is a bilinear pairing $B : V \times V \rightarrow k$. A $k$-bilinear form $B$ is said to be

- **symmetric** provided that $B(x, y) = B(y, x)$ for all $x$ and $y \in V$,
- **alternating** provided that $B(u, u) = 0$ for all $u \in V$, and
- **skew-symmetric** (or anti-symmetric) provided that $B(s, t) = -B(t, s)$ for all $s$ and $t \in V$.

(183) Show that every alternating form is skew symmetric. Hint for this problem and the next two: Think about $B(v + w, v + w)$.

(184) Show that, if the characteristic of $k$ is not 2, then every skew-symmetric form is alternating.

(185) Show that, if the characteristic of $k$ is not 2 and $B$ is a symmetric bilinear form with $B(v, v) = 0$ for all $v \in V$, then $B(v, w) = 0$ for all $v$ and $w \in V$.

We now restrict our attention to the finite dimensional case:

(186) Let $v_1, v_2, \ldots, v_n$ be a basis of $V$ and let $G$ be the $n \times n$ matrix $G_{ij} = B(v_i, v_j)$. We call $G$ the Gram matrix.

(a) In the basis $v_1, \ldots, v_n$, verify the formula $B(\bar{x}, \bar{y}) = \bar{x}^T G \bar{y}$.

(b) Under what conditions on $G$ will $B$ be symmetric?

(c) Under what conditions on $G$ will $B$ be alternating?

(d) Under what conditions on $G$ will $B$ be skew-symmetric?

(187) Let $w_1$, $w_2$, $\ldots$, $w_n$ be a second basis of $V$, with $w_j = \sum S_{ij} v_i$. Let $H$ be the Gram matrix $B(w_i, w_j)$. Give a formula for $H$ in terms of $S$ and $G$.

A bilinear form $B$ on $V$ is called **nondegenerate** if, for all $v \in V$, there is some $w \in V$ with $B(v, w) \neq 0$.

(188) Let $V$ be a finite dimensional vector space. Show that $B$ is nondegenerate if and only if the Gram matrix of $B$ is invertible.

(189) Let $V$ be a finite dimensional vector space, let $B$ be a bilinear form on $V$ and let $L$ be a subspace of $V$ such that the restriction of $B$ to $L$ is nondegenerate. Define $L^\perp = \{ v \in V : B(u, v) = 0 \forall u \in L \}$. Show that $V = L \oplus L^\perp$.

**Note:** In class, I thought that Problem 189 would break if $B$ were not symmetric. In fact, the problem is right as written. However, it is only in the symmetric case that we will have $B(u, v) = 0$ for $u \in L^\perp$ and $v \in L$. In general, let $L^\perp = \{ v \in V : B(u, v) = 0 \forall u \in L \}$ and let $\perp L = \{ v \in V : B(v, u) = 0 \forall u \in L \}$. Then under the hypotheses of Problem 189 it is true both that $V = L \oplus L^\perp$ and that $V = L \oplus \perp L$. However, it is only in the symmetric and skew symmetric cases that we will have $L^\perp = \perp L$.

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1The term Gram matrix is generally used in the context of applied linear algebra, such as computer graphics and control theory. In that context, the vector space $V$ is simply $\mathbb{R}^n$ and $B$ is simply dot product, but $v_i$ is some basis of $\mathbb{R}^n$ which is not orthonormal. The Gram matrix encodes the “skewness” of our basis.

2Without finite dimensionality, this is not true. Let $V$ be a vector space with basis $e_1, e_2, e_3, \ldots$ and consider the standard bilinear form $B(\sum a_i e_i, \sum b_i e_i) = \sum a_i b_i$. Let $L$ be the subspace spanned by $e_i - e_j$. Then $L^\perp$ is 0 because, if $\sum a_k e_k$ is perpendicular to all $e_i - e_j$ then $a_i = a_j$ for all $i, j$. But $V$ only allows finite sums, so the only such element are 0.