**Vocabulary: Product of ideals, comaximal ideals**

This is a good time to toss in the definition of the product of ideals:

**Definition.** Suppose $R$ is a ring. The *product* of ideals $A$ and $B$ in $R$ is the ideal, denoted $AB$, consisting of all finite sums $\sum a_ib_i$ with $(a_i, b_i) \in A \times B$. The product of any finite number of ideals is defined similarly.

We now introduce the notion of comaximal ideal. As we will see, ideals being comaximal is something like integers being relatively prime.

**Definition.** Suppose $R$ is a commutative ring. Ideals $A$, $B$ of $R$ are said to be *comaximal* provided that $A + B = R$.

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1. Show that $A$ and $B$ are comaximal if and only if $1 \in A + B$.
2. If $m$ is maximal and $I$ is an ideal, show that either $m$ and $I$ are comaximal, or else $I \subseteq m$.
3. Give an example of two distinct prime ideals of $k[x, y]$ which are not comaximal.
4. Suppose $n$ and $m$ are integers. Show that $n\mathbb{Z}$ and $m\mathbb{Z}$ are comaximal if and only if $\text{GCD}(n, m) = 1$. You may assume uniqueness of prime factorization for this question.
5. Let $R$ be a commutative ring and let $A$ and $B$ be ideals. Show that the map $R \to R/A \times R/B$, sending $r$ to the ordered pair $(r \mod A, r \mod B)$, is surjective if and only if $A$ and $B$ are comaximal.
6. Suppose that $A$ and $B$ are comaximal ideals in a commutative ring $R$. Show that $A \cap B = AB$.
7. Suppose that $R$ is a nonzero commutative ring. Suppose $I_1, I_2, I_3, \ldots, I_k$ are ideals in $R$ that are pairwise comaximal. Show that the ideals $I_1$ and $I_2I_3 \cdots I_k$ are comaximal.

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1As yet, we have not discussed products of rings, but this problem only considers $R/A \times R/B$ as a set.