Let $B$ be a symmetric bilinear form on a vector space $W$ over $\mathbb{R}$. We say that $B$ is

- **Positive definite** if $B(w, w) > 0$ for all nonzero $w \in W$.
- **Positive semidefinite** if $B(w, w) \geq 0$ for all $w \in W$.
- **Negative definite** if $B(w, w) < 0$ for all nonzero $w \in W$.
- **Negative semidefinite** if $B(w, w) \leq 0$ for all $w \in W$.

Recall that we showed in Problem G.6 that a symmetric bilinear form over $\mathbb{R}$ can always be represented by a diagonal matrix whose entries lie in $\{−1, 0, 1\}$.

**Problem H.1.** Let $B$ be a symmetric bilinear form which can be represented by the diagonal matrix

$$\text{diag}(n_+,\ldots,1,0,\ldots,0,-1,-1,\ldots,-1).$$

(1) Show that $n_+$ is the dimension of the largest subspace $L$ of $V$ such that $B$ restricted to $L$ is positive definite.
(2) Show that $n_+ + n_0$ is the dimension of the largest subspace $L$ of $V$ such that $B$ restricted to $L$ is positive semidefinite.
(3) Show that $n_-$ is the dimension of the largest subspace $L$ of $V$ such that $B$ restricted to $L$ is negative definite.
(4) Show that $n_- + n_0$ is the dimension of the largest subspace $L$ of $V$ such that $B$ restricted to $L$ is negative semidefinite.

**Problem H.2.** Let $B$ be a symmetric bilinear form. Suppose that $B$ can be represented (in two different bases) by the diagonal matrices

$$\text{diag}(m_+,\ldots,1,0,\ldots,0,-1,-1,\ldots,-1) \quad \text{and} \quad \text{diag}(n_+,\ldots,1,0,\ldots,0,-1,-1,\ldots,-1).$$

Show that $(m_+, m_0, m_-) = (n_+, n_0, n_-)$.

The word **signature** is used to refer to something like the triple $(n_+, n_0, n_-)$. Unfortunately, sources disagree as to exactly what the signature is. Various sources will say that the signature is $(n_+, n_0, n_-)$, $(n_+, n_-, n_0)$, $(n_+, n_-)$ or $n_+ - n_-$. In this course, we’ll adopt the convention that the signature is $(n_+, n_0, n_-)$. If $G$ is a symmetric real matrix, we will use the term **signature of $G$** to refer to the signature of the bilinear form $B(x, y) = x^T G y$.

**Problem H.3.**
Let $G$ be a real symmetric $n \times n$ matrix with signature $(n_+, n_0, n_-)$. If $n_0 > 0$, show that $\det G = 0$. If $n_0 = 0$, show that $\det G$ is nonzero with sign $(-1)^{n_-}$.

**Problem H.4.** Let $G$ be a real symmetric $n \times n$ matrix with signature $(n_+, n_0, n_-)$. Let $G'$ be the upper left symmetric $(n-1) \times (n-1)$ submatrix of $G$. Show that the signature of $G'$ is one of $(n_+ - 1, n_0 + 1, n_- - 1)$, $(n_+ - 1, n_0, n_-)$, $(n_+, n_0, n_- - 1)$, $(n_+, n_0 - 1, n_-)$. Hint: Use Problem H.1.

**Problem H.5.** Let $G$ be a real symmetric matrix and let $G_k$ be the $k \times k$ upper left submatrix of $G$. Assume that $\det G_k \neq 0$ for $1 \leq k \neq n$. Show that the signature of $G$ is $(n - q, 0, q)$ where $q$ is the number of $k$ for which $\det G_{k-1}$ and $\det G_k$ have opposite signs. Here we formally define $\det G_0 = 1$.

**Problem H.6.** (Sylvester’s criterion) Let $G$ be a real symmetric matrix and define $G_k$ as above. Show that $G$ is positive definite if and only if all the $\det G_k$ are $> 0$. (In other words, we no longer have to take $\det G_k \neq 0$ as an assumption.)