## IDEALS

**Vocabulary:** Ideal, left ideal, right ideal, two-sided ideal, cyclic module, quotient ring

### Definition
Suppose $R$ is a ring. A subset $I \subset R$ is called a **left ideal** provided that

1. $(I, +)$ is a subgroup of $(R, +)$; and
2. for all $r \in R$ we have $rI \subset I$, that is $rx \in I$ for all $x \in I$.

It is called a **right ideal** provided that

1. $(I, +)$ is a subgroup of $(R, +)$; and
2. for all $r \in R$ we have $Ir \subset I$, that is $yr \in I$ for all $y \in I$.

A subset of $R$ that is both a left and right ideal is called an **ideal** or **two-sided ideal**.

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*Professor Speyer’s usual practice is to always say “two-sided ideal” in a non-commutative ring. In his experience, this is common among mathematicians who usually work in the commutative world, but who need to deal with non-commutative things occasionally.*

If $R$ is commutative, then every left ideal is a right ideal is a two-sided ideal is an ideal.

(20) Show that if $A$ and $B$ are ideals, then $A + B := \{a + b : a \in A, b \in B\}$ is also an ideal.

(21) Let $D$ be a nonsquare integer, let $R = \mathbb{Z}[\sqrt{D}]$ and let $p$ be a prime. Show that $R$ has an ideal $I$ with $|R/I| = p$ if and only if $D$ is a square modulo $p$.

(22) Fix $n \geq 2$. Let $I$ be the subset of $R = \text{Mat}_{n \times n}(\mathbb{Q})$ consisting of matrices with nonzero entries only in the first row. Is $I$ a left ideal? Is it a right ideal?

(23) Suppose $R$ and $S$ are rings and $\varphi \in \text{Hom}(R, S)$. Show that $\ker(\varphi)$ is a two-sided ideal of $R$.

(24) Let $R$ be a ring and let $I$ be a left ideal. Since $I$ and $R$ are abelian groups with respect to $+_R$, we can form the quotient group $R/I$. Show that $R/I$ has a natural structure as a left $R$-module.

An $R$-module which can be written in the form $R/I$ is called **cyclic**.

(25) Let $R$ be a ring and let $I$ be a two sided ideal. Show that $R/I$ has a natural ring structure.