Classification of finitely generated modules over a PID

Throughout this worksheet, let \( R \) be a PID.

(103) Let \( X \in \text{Mat}_{m \times n}(R) \) and let \((d_1, d_2, \ldots, d_{\min(m,n)})\) be the invariant factors of \( X \).

(a) Show that \( R^m/XR^n \cong \bigoplus R/d_jR \oplus R^{m-\min(m,n)} \).

(b) Show that \( \text{Ker}(X) \cong R^{\#\{j:d_j=0\}+n-\min(m,n)} \).

(104) Let \( S \) be a ring and let \( M \) be a finitely generated \( S \)-module.

(a) Show that there is a surjection \( S^m \twoheadrightarrow M \) for some \( m \).

(b) Suppose that \( S \) is Noetherian (for example, every PID is Noetherian). Show that there is a surjection \( S^n \twoheadrightarrow \text{Ker}(S^m \twoheadrightarrow M) \) for some \( n \).

(c) With hypotheses and assumptions as in the previous part, show that there is an \( m \times n \) matrix \( X \) with \( M \cong S^m/XS^n \).

(105) **Classification of modules over a PID: Elementary divisor form**

Show that every finitely generated \( R \)-module \( M \) is of the form \( \bigoplus R/d_jR \) for some nonunits \( d_1, d_2, \ldots, d_k \) in \( R \) with \( d_1 \mid d_2 \mid \cdots \mid d_k \).

(106) **Classification of modules over a PID: Prime power form**

Show that every finitely generated \( R \)-module \( M \) is of the form \( R^\oplus \bigoplus R/p_j^{e_j}R \) for some nonnegative integer \( r \), some sequence of prime elements \( p_j \) and some sequence of positive integers \( e_j \).

(107) Let \( M \) be a finitely generated \( R \)-module.

(a) Show that the \( d_1, d_2, \ldots, d_k \) in Problem 105 are unique up to multiplication by units.

(b) Show that the \( r, p_j \) and \( e_j \) in Problem 106 are unique up to rearrangement and up to multiplying the \( p_j \) by units.

Hint for both parts: One approach is to study \( M/qM \) for various choices of \( q \).

Let's see what this say for some particular PID's:

(108) Let \( k \) be a field, then \( k \) is also a PID. What have we proved about finitely generated \( k \)-modules?

(109) A \( \mathbb{Z} \)-module is the same thing as an abelian group.

(a) What have we proved about finitely generated abelian groups?

(b) Consider the matrices below as maps \( \mathbb{Z}^n \to \mathbb{Z}^m \). Describe their cokernels:

\[
\begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}
\quad \begin{bmatrix}
2 & 1 \\
0 & 2
\end{bmatrix}
\quad \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix}.
\]