Problem Set 1: (Due Friday September 13)

Please see the course website for policy regarding collaboration and formatting your homework.

1. Suppose $R$ is a ring and $a, b \in R$. Show that, if $b$ is a unit and $a$ divides $b$ on both the left and the right, then $a$ is a unit.

2. Let $K$ be a field and let $R$ be a subring of $K$. Let $S$ be a nonempty subset of $R$, closed under multiplication and not containing $0$. Let $S^{-1}R$ be the set of elements in $K$ which can be written as $\frac{a}{b}$ with $a \in R$ and $b \in S$. Show that $S^{-1}R$ is a subring of $K$.

3. Let $A$ be a ring. The center of $A$ is $Z(A) := \{z \in A : az = za \text{ for all } a \in A\}$. Show that $Z(A)$ is a subring of $A$.

4. What is the cardinality of the following rings?
   (a) $\mathbb{Z}[x]/(6, 2x - 1)$.
   (b) $\mathbb{Z}[x]/(x^2 - 3, 2x + 4)$.

5. Let $k$ be a field. Show that the only ideals of $k$ are $(0)$ and $k$.

6. Let $k$ be a field.
   (a) Describe all left ideals of in the ring $\text{Mat}_{n \times n}(k)$ of $n \times n$ matrices with entries in $k$. Hint: Row reduction.
   (b) Show that the only two-sided ideals of $\text{Mat}_{n \times n}(k)$ are $(0)$ and $\text{Mat}_{n \times n}(k)$.

7. Let $R$ be the set of all infinite sequences $(x_1, x_2, \ldots)$ in $\mathbb{R}$ for which $\lim_{n \to \infty} x_n$ exists. We define addition and multiplication on $R$ by $(x_j) + (y_j) = (x_j + y_j)$ and $(x_j)(y_j) = (x_jy_j)$.
   (a) Show that $R$ is a commutative ring.
   (b) Let $m$ be the set of sequences $(x_j)$ for which $\lim_{n \to \infty} x_n = 0$. Show that $m$ is a maximal ideal of $R$.

8. Suppose $k$ is a field and let $\bar{R} = k[t]$. Show that $R$-modules are “the same” as $k$-vector spaces $V$ equipped with a $k$-linear endomorphism $T: V \to V$. This question can be interpreted in two ways:
   • (for those who don’t know what categories are) Give a bijection between isomorphism classes of $R$-modules and isomorphism classes of pairs $(V, T)$; this includes defining when $(V_1, T_1)$ and $(V_2, T_2)$ are isomorphic.
   • (for those who know what categories are) Define the category of $R$-modules and the category of pairs $(V, T)$, and give an equivalence between them.

9. Suppose $k$ is a commutative ring. An $k$-algebra is an $k$-module $A$ with an $k$-bilinear map $A \times A \to A$.
   Let $k$ be a commutative ring and let $A$ be any ring. Give a bijection between ways to consider $A$ as an $k$-algebra, and ring maps $k \to Z(A)$. (This problem includes defining what “ways to consider $A$ as an $k$-algebra” means.)

10. Suppose $R$ is a ring. An element $e \in R$ is called idempotent if $e^2 = e$.
    (a) Give an example of an idempotent, other than $0$ and $1$, in $\text{Mat}_{2 \times 2}(\mathbb{Z})$.
    (b) Give an example of an idempotent, other than $0$ and $1$, in $\mathbb{Z}/15\mathbb{Z}$.
    (c) Let $e$ be an idempotent of $R$ and let $eRe = \{ere : r \in R\}$. Show that $eRe$ is a ring, with respect to the addition and multiplication operations of $R$, where $0_{eRe} = 0_R$ and $1_{eRe} = e$.

11. Let $R$ be a ring. For $r \in R$ and $1 \leq i \neq j \leq n$ define the $n \times n$ matrix $E(i, j, r)$ by

   $$E(i, j, r)_{k \ell} = \begin{cases} 1 & \text{if } k = \ell, \\ r & \text{if } k = i \text{ and } j = \ell, \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

   The matrix $E(i, j, r)$ is known as an elementary matrix.
    (a) Suppose $X$ is an $m \times n$ matrix. What is the effect of right multiplication by $E(i, j, r)$ on $X$? Suppose $Y$ is an $n \times m$ matrix. What is the effect of left multiplication by $E(i, j, r)$ on $Y$? What is the inverse of $E(i, j, r)$?
    (b) Show that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a product of $2 \times 2$ elementary matrices.
    (c) Let $u$ be a unit of $R$. Show that the matrix $\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$ is a product of $2 \times 2$ elementary matrices.

Additional task: We will use Zorn’s Lemma in this class. Please familiarize yourself with its statement and with how it is used to show that every nonzero ring has a maximal ideal. Good references are pages 907-909 in Dummit and Foote, Keith Conrad’s notes [https://kconrad.math.uconn.edu/blurbs/zorn1.pdf](https://kconrad.math.uconn.edu/blurbs/zorn1.pdf) and Dan Grayson’s proof at [https://faculty.math.illinois.edu/~dan/ShortProofs/Zorn.pdf](https://faculty.math.illinois.edu/~dan/ShortProofs/Zorn.pdf).