Please see the course website for policy regarding collaboration and formatting your homework.

(20) Let $R$ be a commutative ring and $P$ a prime ideal such that $R/P$ is finite. Show that $P$ is maximal.

(21) Let $R$ be a finite ring, and let $|R|$ factor as $\prod p^{n_p}$. Show that there are rings $R_{p_i}$, for the various primes $p$ dividing $N$, such that $|R_{p_i}| = p^{n_p}$ and $R \cong \prod R_{p_i}$.

(22) As promised, we revisit Problem (14) without the hypothesis that the elements of $S$ are not zero divisors. Let $R$ be a commutative ring and let $S$ be a subset of $R$ which is closed under multiplication and contains 1. Define a relation $\sim$ on $S \times R$ by $(s, r) \sim (s', r')$ if there is an element $s''$ of $S$ such that $s''sr' = s''s'r$.

(a) Show that $\sim$ is an equivalence relation.

(b) Show that the simpler definition, $(s, r) \sim (s', r')$ if $sr' = s'r$, need not be an equivalence relation.

As before, we define $S^{-1}R$ to be $S \times R/\sim$. I won’t make you write it out but, once again, $S^{-1}R$ is a ring.

(c) Give a simple description of $\{3^k : k \in \mathbb{Z}_{\geq 0}\}^{-1}(\mathbb{Z}/15\mathbb{Z})$.

(d) Show that the kernel of the map $r \mapsto 1^{-1}r$ from $R$ to $S^{-1}R$ is $\{x \in R : \exists s \in S \text{ sx = 0}\}$.

(e) State and prove a universal property of $S^{-1}R$. In other words, your statement should look like “Given a commutative ring $R'$, and the following additional data . . . , there is a unique map $S^{-1}R \to R'$ such that . . . ”

(23) Let $R$ be a commutative ring and let $P$ be a prime ideal of $R$.

(a) Show that $R \setminus P$ is closed under multiplication.

We define $R_P := (R \setminus P)^{-1}R$ and call $R_P$ the local ring of $P$.

(b) Justify this terminology by showing that $R_P$ is a local ring (see problem 15).

(24) An element $x$ in a ring $R$ is called nilpotent if there is a positive integer $N$ such that $x^N = 0$.

(a) Show that, if $x$ is nilpotent, then $1 + x$ is a unit.

(b) Show that, if $x$ and $y$ are nilpotents with $xy = yx$, then $x + y$ is nilpotent. Is this true if $x$ and $y$ do not commute?

(c) Show that if $R$ is commutative, then the set of nilpotent elements in $R$ form an ideal; it is called the nilradical of $R$ and often denoted $\text{Nil}(R)$.

(25) Let $R$ be a UFD.

(a) Let $A$ and $B$ be matrices with entries in $R$, of sizes $r \times s$ and $s \times t$ respectively, and let $C = AB$; we write $A_{ij}, B_{jk}$ and $C_{ik}$ for the entries of these matrices. Prove or disprove: $\text{GCD}(C_{ik}) = \text{GCD}(A_{ij}) \text{ GCD}(B_{jk})$.

(b) Let $a(x) = \sum a_ix^i$ and $b(x) = \sum b_ix^i$ be polynomials with coefficients in $R$ and let $c(x) = a(b(x)) = \sum c_ix^i$. Prove or disprove: $\text{GCD}(c_k) = \text{GCD}(a_i) \text{ GCD}(b_j)$.

(26) Show that the following rings are Noetherian:

(a) The ring of integers, $\mathbb{Z}$.

(b) Any ring which is finite dimensional as a $k$ vector space, for a field $k$.

(c) The polynomial ring $k[x]$, for any field $k$.

(27) Show that the following rings are not Noetherian:

(a) The polynomial ring $k[x_1, x_2, \ldots]$ in infinitely many variables.

(b) $k[x, x^{1/2}, x^{1/4}, x^{1/8}, \ldots]$, for $k$ any field. An element of this ring is a formal finite sum $\sum a_jx^{1/2^j}$.

(c) The subring of $k[x, y]$ generated by all monomials of the form $x^iy^j$.

(28) Let $R$ be a Noetherian ring and let $I$ be a two-sided ideal. Show that $R/I$ is Noetherian.

(29) This problem provides a proof of Hilbert’s basis theorem, which states: If $R$ is a Noetherian commutative ring, then $R[x]$ is Noetherian. Problems 26c, 26a and 28, together with Hilbert’s basis theorem, show that any commutative ring which is finitely generated over $k$ or $\mathbb{Z}$ is Noetherian.

Let $I$ be an ideal of $R[x]$; we will show that $I$ is finitely generated. Define $I_d$ to be the set of $f \in R$ such that there is an element of $I$ of the form $fx^d + f_{d-1}x^{d-1} + \cdots + f_1t + f_0$.

(a) Show that $I_d$ is an ideal of $R$ and that $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$.

(b) Show that there is an index $r$ such that $I_r = I_{r+1} = I_{r+2} = \cdots$. Show that $I_r$ is finitely generated.

(c) Let $M$ be the set of polynomials in $f$ with degree $\leq r$. Show that $M$ is finitely generated as an $R$-module. Let $f_1, f_2, \ldots, f_m$ generate $I_r$ as an $R$-module and choose elements $g_j$ of $I$ of the form $g_j = f_jx^r + $ (lower order terms). Let $h_1, h_2, \ldots, h_n$ generate $M$ as an $R$-module.

(d) Show that $g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_n$ generate $I$ as an $R[x]$ module.