Problem Set Six: Due October 27

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 6.

Problem 2. Write up complete solutions to three two of the following problems from class: 13.5, 13.7, 13.8, 14.5, 14.8, 16.5

Problem 3. In the ring $\mathbb{Z}[x]$, is 15 a unit, irreducible or a composite? What about in the ring $\mathbb{Q}[x]$?

Problem 4. (1) Compute the GCD of 2021 and 215 using the Euclidean algorithm.
(2) Find integers $x$ and $y$ such that $2021x + 215y = \text{GCD}(2021, 215)$.

Problem 5. Let $k$ be a field and let $b_1(t), b_2(t), \ldots, b_r(t)$ be pairwise relatively prime polynomials in $k[t]$. Set $g(t) = b_1(t)b_2(t) \cdots b_r(t)$.

1. Show that the polynomials $g(t)/b_j(t)$ generate $k[t]/g(t)k[t]$ as a $k[t]$-module.
2. Let $f(t) \in k[t]$. Show that there are polynomials $a_1(t), \ldots, a_r(t)$ and $c(t)$ in $k[t]$ such that
   \[ f(t) g(t) = \sum_{j=1}^{r} a_j(t) \frac{g(t)}{b_j(t)} + c(t). \]

   Now you know why integration by partial fractions works!

Problem 6. Let $R$ be a PID and let $x$ be a nonzero element of $R$. Let $M = R/xR$ and let $y$ be another nonzero element of $R$.

1. Show that $yM \cong R/\text{GCD}(x, y)R$ as $R$-modules.
2. Give similar descriptions for the $R$-modules $M/yM$ and $M[y] := \{m \in M : ym = 0\}$.
3. Show that $M$ has finite length as an $R$-module, and describe $\ell(M)$ in terms of a prime factorization of $x$.

Problem 7. Let $R$ be a PID.

1. Let $x$ and $y \in R$ with $\text{GCD}(x, y) = g$. Show that there is a matrix $[\begin{array}{cc} a & b \\ c & d \end{array}]$ with entries in $R$ such that $ad - bc = 1$ and $[\begin{array}{c} a \\ c \end{array}][\begin{array}{c} x \\ y \end{array}] = [\begin{array}{c} g \\ 0 \end{array}]$.
2. Let $x_1, x_2, \ldots, x_n \in R$ with $\text{GCD}(x_1, x_2, \ldots, x_n) = g$. Show that there is an $n \times n$ matrix $A$ with entries in $R$ such that $\det A = 1$ and $A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T = [g \ 0 \ \cdots \ 0]^T$.
3. Let $x$ and $y \in R$. Show that there are invertible $2 \times 2$ matrices $U$ and $V$ with
   \[ U \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} V = \begin{bmatrix} \text{GCD}(x, y) & 0 \\ 0 & \text{LCM}(x, y) \end{bmatrix}. \]

Problem 8. Let $R$ be a commutative ring. Let $A$ and $B$ be $m \times n$ matrices with entries in $R$ and suppose that there exist invertible matrices $U$ and $V$, of sizes $m \times m$ and $n \times n$ such that $B = UAV$.

1. Show that the ideal generated by all the entries of $A$ is the same as the ideal generated by all the entries of $B$.
2. Take $R = \mathbb{Q}[x, y]$, $A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & xy \end{bmatrix}$. Show that there do not exist invertible matrices $U$ and $V$ with $B = UAV$.
3. With $R$, $A$ and $B$ as above, are the modules $R^2/AR^2$ and $R/BR^2$ isomorphic or not?

Problem 9. Let $R$ be an integral domain and let $p_1, p_2, \ldots, p_k$ be elements of $R$ such that $p_1R, p_2R, \ldots, p_kR$ are distinct maximal ideals. Let $I$ be an ideal containing $p_1p_2 \cdots p_k$. Show that $I$ is $p_1p_2 \cdots p_sR$ for some subset $\{i_1, i_2, \ldots, i_s\}$ of $\{1, 2, \ldots, k\}$.