Problem Set 7: Due November 3

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 7.

Problem 2. Please write up proofs of the following class problems: 17.5, and either 18.1 or 18.4.

Problem 3. (1) Let \( A = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} \). Write \( A \) in Smith normal form.

(2) Let \( B = \begin{bmatrix} 2 & 4 & 10 \\ 1 & 3 & 7 \\ 1 & 1 & 15 \end{bmatrix} \). If we were to write \( B \) in Smith Normal form as \( UDV \), what would \( D \) be? (You need not find \( U \) and \( V \).)

Problem 4. (1) Use the Euclidean algorithm to find polynomials \( f(t) \) and \( g(t) \) in \( \mathbb{Q}[t] \) with
\[
f(t)(3t^2 - 3t - 1) + g(t)(t^3 - 2) = 1.
\]

(2) Find rational numbers \( a, b, c \) such that
\[
(3\sqrt{4} - 3\sqrt{2} - 1)^{-1} = a\sqrt{4} + b\sqrt{2} + c.
\]
Now you can rationalize denominators for algebraic numbers of degree greater than 2!

Problem 5. Let \( R \) be an integral domain and let \( I \) be a nonzero ideal of \( R \).

(1) Draw arrows indicating which implications exist between the following concepts. You need not provide proofs or counterexamples:

<table>
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<tr>
<th>( I ) is prime</th>
<th>( I ) is maximal</th>
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<tr>
<td>( I ) is of the form ( (f) ) for ( f ) irreducible</td>
<td>( I ) is of the form ( (f) ) for ( f ) prime</td>
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(2) How would your answers change if we assume that \( R \) is a UFD?

(3) How would your answers change if we assume that \( R \) is a PID?

Problem 6. Let \( R \) be a commutative ring and let \( A \) be an \( n \times n \) matrix with entries in \( R \). The point of this problem is to prove that the following are equivalent:

(a) \( \det A \) is a unit of \( R \).

(b) There is an \( n \times n \) matrix \( B \) with \( AB = \text{Id}_n \).

(c) There is an \( n \times n \) matrix \( C \) with \( CA = \text{Id}_n \).

For ease of grading, break up your proof as follows:

(1) Show that (b) or (c) implies (a).

(2) Show that (a) implies (b) and (c). Hint: Look up the adjugate matrix.

(3) Show that, if (b) and (c) hold, then \( B = C \).

Problem 7. The following problem gives a criterion for a ring to be a PID which is similar to, but more general, than being Euclidean. Let \( R \) be an integral domain and let \( N(\ ) \) be a positive norm on \( R \). Let \( P \) be a subset of \( R \) such that, for all \( a \) and \( b \in R \) with \( b \neq 0 \), there are \( p \in P \) and \( q \) and \( r \in R \) such that \( pa = qb + r \) and \( N(r) < N(b) \).

Let \( J \) be a nonzero ideal of \( R \) and let \( b \) be a nonzero element of \( J \) of minimal norm.

(1) Show that, for every \( a \in J \), there is some \( p \in P \) with \( pa \in bR \).

(2) Now suppose that \( R \) is Noetherian. Show that there are distinct elements \( p_1, p_2, \ldots, p_k \) of \( R \) with \( \prod p_i \mid J \subseteq bR \).

(3) Suppose that \( R \) is Noetherian and that \( pR \) is maximal for all \( p \in P \). Show that \( J \) is principal. You may use Problem 9 from Problem Set 6 even if you haven’t solved it.