Problem Set 8 (Due Friday, November 15)

Please see the course website for policy regarding collaboration and formatting your homework.

(60) Consider the nilpotent matrix whose powers are shown below:

\[
X = \begin{bmatrix}
-1 & -7 & -2 \\
0 & 3 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

\[
X^2 = \begin{bmatrix}
-1 & -4 & -1 \\
-2 & 4 & -2 \\
-1 & -4 & -1
\end{bmatrix}
\]

\[
X^3 = 0
\]

\[
\text{rank}(X) = 2, \quad \text{rank}(X^2) = 1, \quad \text{rank}(X^3) = 0
\]

(a) Compute the Jordan normal form of \(X\). Suppose that \(S\) is a matrix with columns \(v_1, v_2, v_3, v_4\) such that \(S^{-1}XS\) is in Jordan normal form, with the larger blocks sorted before the smaller ones. Express the following quantities in terms of \(v_1, v_2, v_3, v_4\):

(b) \(\text{Ker}(X)\) and \(\text{Ker}(X^2)\).

(c) \(\text{Im}(X)\) and \(\text{Im}(X^2)\).

(d) The intersections \(\text{Im}(X) \cap \text{Ker}(X), \text{Im}(X) \cap \text{Ker}(X^2), \text{Im}(X^2) \cap \text{Ker}(X)\) and \(\text{Im}(X^2) \cap \text{Ker}(X^2)\).

(61) Let \(A\) be an \(n \times n\) complex matrix. Show that \(A\) is nilpotent if and only if \(A\) and \(2A\) are similar.

(62) Suppose that \(A\) is a \(7 \times 7\) complex matrix that obeys the polynomial relation \(A^5 = 2A^4 + A^3\). Given that the rank of \(A\) is 5 and the trace is 4, what is the Jordan canonical form of \(A\)?

(63) We didn’t get to this one in class, so please do it now. Let \(K \subseteq L\) be fields and let \(X\) be an \(n \times n\) matrix with entries in \(K\). Show that \(X\) has the same rational canonical form over \(K\) and over \(L\).

(64) Let \(k\) be a field. A square matrix \(X\) with entries in \(k\) is called \textbf{diagonalizable} if it is similar to a diagonal matrix.

(a) Show that, if a matrix \(X\) is both diagonalizable and nilpotent, then it is zero.

(b) Let \(X\) and \(Y\) be square matrices with entries in \(k\) and let \(Z\) be the block matrix \(
\begin{bmatrix}
X & 0 \\
0 & Y
\end{bmatrix}
\)

Show that \(Z\) is diagonalizable if and only if \(X\) and \(Y\) are both diagonalizable.

(c) Let \(X\) and \(Y\) be diagonalizable square matrices which commute with each other. Show that there is an invertible matrix \(S\) such that \(SXS^{-1}\) and \(SYS^{-1}\) are both diagonal. Hint: Start by reducing to the case that \(X\) is diagonal and remember to use the hypothesis that \(XY = YX\).

(65) Let \(k\) be an algebraically closed field\(^1\) and let \(X\) be an \(n \times n\) matrix with entries in \(k\).

(a) Show that \(X\) can be written in the form \(D + N\) where \(D\) is diagonalizable, \(N\) is nilpotent, and \(DN = ND\).

(b) Let \(\lambda_i\) be the eigenvalues of \(X\), choose \(B \geq n\) and let \(g(x)\) be the polynomial in Problem\(^5\). Show that the \(D\) you have constructed is \(g(X)\).

The decomposition \(X = D + N\) is known as the Jordan-Chevalley decomposition of \(X\). It is unique, and we will likely prove so on a future problem set.

(66) This problem is a sequel to Problem\(^5\). You may use any results from that problem without proof. We recall the key definitions: Let \(R\) be a ring. A left \(R\)-module \(S\) is called simple if \(S \neq 0\) and \(S\) has no submodules other than 0 and \(S\). A left \(R\)-module \(M\) is said to have \textbf{finite length} if there is a chain of submodules \(0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\ell = M\) such that \(M_{i+1}/M_i\) is simple for \(0 \leq i < \ell\).

Let \(M\) be a module of finite length with a chain \(0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\ell = M\) as above.

(a) Show that \(M\) does not contain an infinite chain \(V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \cdots \) of \(R\)-submodules, nor does it contain an infinite chain \(W_1 \subsetneq W_2 \subsetneq W_3 \subsetneq \cdots \) of \(R\)-submodules. Hint: Induct on \(\ell\), and think about \(0 \rightarrow M_j \rightarrow M \rightarrow M/M_j \rightarrow 0\).

Let \(\phi : M \rightarrow M\) be an \(R\)-module homomorphism.

(b) Show that, for \(N\) sufficiently large, we have \(\text{Ker}(\phi^N) = \text{Ker}(\phi^{N+1}) = \text{Ker}(\phi^{N+2}) = \cdots\). Call this common kernel \(\hat{K}\). Show furthermore that \(\phi\) takes \(\hat{K}\) to \(\hat{K}\) and that the restriction of \(\phi\) to \(\hat{K}\) is nilpotent.

(c) Show that, for \(N\) sufficiently large, we have \(\text{Im}(\phi^N) = \text{Im}(\phi^{N+1}) = \text{Im}(\phi^{N+2}) = \cdots\). Call this common image \(\hat{I}\). Show furthermore that \(\phi\) takes \(\hat{I}\) to \(\hat{I}\) and that the restriction of \(\phi\) to \(\hat{I}\) is invertible.

(d) Show that \(M = \hat{K} \oplus \hat{I}\).

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\(^1\)Jordan-Chevalley decomposition can actually be defined for perfect fields, but “diagonalizable” must be replaced by a more subtle condition called “semi-simple”. Here a field \(k\) is perfect if either (1) \(k\) has characteristic 0 or (2) \(k\) has characteristic \(p\) and all elements of \(k\) have \(p\)-th roots.