Problem Set 9 (Due Friday, November 22)

(67) Let \( k \) be a field, let \( V \) be an \( n \)-dimensional \( k \)-vector space and let \( \alpha : V \to V \) be a diagonalizable map with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \), each of multiplicity 1. Compute the eigenvalues of \( \alpha \otimes \alpha \).

(68) Recall the definition of the Jordan block \( J_n(\lambda) \). Compute the Jordan canonical form of \( J_2(0) \otimes J_3(0) \) as a map from \( \mathbb{R}^2 \otimes \mathbb{R}^3 \) to itself.

(69) Let \( k \) be a field and let \( V \) and \( W \) be vector spaces with bases \( e_1, e_2, \ldots, e_m \) and \( f_1, f_2, \ldots, f_n \). Let \( \tau = \sum t_{ij} e_i \otimes f_j \) be an element of \( V \otimes W \). Show that we can write \( \tau \) in the form \( v_1 \otimes w_1 + v_2 \otimes w_2 + \cdots + v_r \otimes w_r \) if and only if the matrix \( [t_{ij}] \) has rank \( \leq r \).

(70) Let \( k \) be a commutative ring. Let \( R \) be a unital associative \( k \)-algebra and let \( M \) be a \( k \)-module. Construct a “natural” left \( R \)-module structure on \( R \otimes_k M \).

(71) Let \( R \) be a commutative ring and let \( S \) be a subset of \( R \), containing 1 and closed under multiplication. Let \( M \) be an \( R \)-module. Define \( S^{-1}M \) to consist of formal symbols \( s^{-1}m \) with \( s \in S \) and \( m \in M \), modulo the relation that \( s_1^{-1}m_1 \equiv s_2^{-1}m_2 \) if there is some \( s_3 \in S \) such that \( s_2s_3m_1 = s_1s_3m_2 \). You may assume that this is an equivalence relation. We make \( S^{-1}M \) into an \( S^{-1}R \)-module by defining \( s_1^{-1}m_1 + s_2^{-1}m_2 = (s_1s_2)^{-1}(s_2m_1 + s_1m_2) \) and \( (s_1r)(s_2^{-1}m) = (s_1s_2)^{-1}(rm) \). You may assume that this is well defined or that it makes \( S^{-1}M \) into an \( S^{-1}R \)-module.

(a) Show that \( S^{-1}M \cong S^{-1}R \otimes_R M \). Prove this isomorphism at least as \( R \)-modules, and ideally as \( S^{-1}R \)-modules in the sense of Problem 70.

(b) Give an example of a triple \((R, S, M)\) where \( R \) is an integral domain and the map \( m \mapsto 1^{-1}m \) from \( M \) to \( S^{-1}M \) is not injective.

(c) Suppose that \( N \) is an \( R \)-module and \( M \) is an \( R \)-submodule of \( N \). Show that the natural map \( S^{-1}M \to S^{-1}N \) is injective.

(72) We did this problem in class, but it was sketchy enough that I think it is worthwhile to make you redo it. Let \( R \) be a PID. Let \( M \) be an \( R \)-module with \( M \cong \bigoplus R/p_j^{e_j} \oplus R^{m \cdot n} \) for each \( p_j \) is prime and \( e_j \in \mathbb{Z}_{>0} \).

(a) Let \( \pi \) be a prime element of \( R \) and let \( k \) be the field \( R/\pi R \). Compute the dimension of \( \pi^k M/\pi^{k+1}M \) as a \( k \)-vector space.

(b) Show that \( \bigoplus R/p_j^{e_j} \oplus R^{m \cdot n} \cong \bigoplus R/q_j^{f_j} \oplus R^{n \cdot s} \) where the \( q_j \) are prime elements and the \( f_j \) are positive integer exponents. Show that \( r = s \) and that there is some permutation \( \sigma \) and some list of units \( u_j \) such that \( q_j = u_j p_{\sigma(j)} \) and \( f_j e_{\sigma(j)} \).

(73) Let’s prove that a real symmetric matrix is diagonalizable! In this problem, you may assume that the irreducible polynomials in \( \mathbb{R}[x] \) are (1) the linear polynomials and (2) the quadratics \( ax^2 + bx + c \) with \( b^2 - 4ac < 0 \).

(a) Let \( X \) be an \( n \times n \) real matrix and suppose that \( X \) is not diagonalizable. Prove that there is a two dimensional subspace \( V \) of \( \mathbb{R}^n \) such that \( X \) takes \( V \) to itself by a matrix of either the form \( \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \) or \( \begin{bmatrix} 0 & -c \\ 1 & 2\lambda \end{bmatrix} \) with \( b^2 < 4c \).

(b) Show that \( \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} \) is similar to \( \begin{bmatrix} 0 & -c^2 \\ 1 & 2\lambda \end{bmatrix} \). Deduce that we may modify the conclusion of the previous part to say that there is a two dimensional subspace \( V \) of \( \mathbb{R}^n \) such that \( X \) takes \( V \) to itself by a matrix of the form \( \begin{bmatrix} 0 & -c \\ 1 & -b \end{bmatrix} \) with \( b^2 \leq 4c \).

(c) Now suppose that \( X \) is symmetric. Let \( \cdot \) be the ordinary dot product on \( \mathbb{R}^n \). Show that, for any \( v \) and \( w \) in \( \mathbb{R}^n \), we have \( (Xv) \cdot w = v \cdot (Xw) \).

(d) Now suppose that \( X \) is symmetric and non-diagonalizable. Let \( v, w \) be a basis of \( V \) in which \( X \) acts by the matrix \( \begin{bmatrix} 0 & -c \\ 1 & -b \end{bmatrix} \) with \( b^2 \leq 4c \). Show that \( w \cdot w + b(v \cdot w) + c(v \cdot v) = 0 \).

(e) Deduce a contradiction.

(74) This problem is a follow up to Problems 64 and 65. You may use the results from those problems without proof. Let \( k \) be an algebraically closed field and let \( X \) be an \( n \times n \) matrix with entries in \( k \). In Problem 65 you constructed a diagonalizable matrix \( D \) and a nilpotent matrix \( N \) such that \( X = D + N \). Now, suppose we had a second decomposition \( X = D_2 + N_2 \) with \( D_2N_2 = N_2D_2 \) such that \( D_2 \) is diagonalizable and \( N_2 \) is nilpotent.

(f) Show that \( D_2X = XD_2 \) and \( N_2X = XN_2 \). Show that \( D_2D = DDX_2, D_2N = NDX_2, N_2D = DNX_2 \) and \( N_2N = N_2N_2 \).

(g) Show that \( D - D_2 \) is diagonalizable. Hint: Look at Problem 64.

(h) Show that \( N - N_2 \) is nilpotent.

(i) Show that \( D - D_2 = N - N_2 = 0 \).