**Prime and Maximal Ideals in Commutative Rings**

**Vocabulary:** prime, maximal, zero divisor

A ring which can be written in the form $R/I$ is called a *quotient ring* of $R$.

**Definition.** Suppose $R$ is a commutative ring. An ideal $P$ of $R$ is called prime if,

- for all $a$ and $b$ in $R$, if $ab \in P$ then $a \in P$ or $b \in P$.
- The ideal $P$ is not all of $R$.

a) We will discuss prime and maximal ideals in non-commutative rings later.

b) This second condition may seem *ad hoc*; it is a good idea for the same reason that we should define 1 not to be a prime number.

(26) An element $a$ of a commutative ring $S$ is called a zero divisor if there is some $x \neq 0$ in $S$ for which $ax = 0$. Let $R$ be a commutative ring and let $I$ be an ideal of $R$. Show that $I$ is prime if and only if $R/I$ has no nonzero zero divisors, but does have some nonzero element.

(27) What are the prime ideals in $\mathbb{Z}$? You may assume uniqueness of prime factorization for this question.\(^1\)

**Definition.** Suppose $R$ is a commutative ring. An ideal $m$ of $R$ is called maximal if,

- for all $a$ in $R$, if $a \notin m$ then there is some $b \in R$ such that $ab \equiv 1 \pmod{m}$.
- The ideal $m$ is not all of $R$.

(28) Let $R$ be a commutative ring and let $I$ be an ideal of $R$. Show that $I$ is maximal and only if $R/I$ is a field.

(29) Show that a maximal ideal is prime.

(30) Show that an ideal $I \subset R$ is maximal if and only there does not exist an ideal $J$ with $I \subset J \subset R$.

Problem (30) is the motivation for the word “maximal”. Using Zorn’s lemma, and Problem (30), it is easy to show that every ideal in a nonzero commutative ring is contained in a maximal ideal.

(31) Let $R = \mathbb{R}[x, y]$. Show that $yR$ is prime but not maximal.

(32) What are the maximal ideals of $\mathbb{Z}$?

\(^1\) In a week or so, we will discuss unique factorization in commutative rings in general. At that point, we will prove it for $\mathbb{Z}$. The careful student can check that there is no circularity; the problems where I permit you to use it now will not feed into our proof then.