TENSOR ALGEBRAS, SYMMETRIC AND EXTERIOR ALGEBRAS

For this worksheet, we move back to the world of vector spaces. It is possible to study these concepts over a general commutative ring, but this seems like enough for now.

Let \( v \) be a field and let \( V \) be a vector space over \( k \). By problem \([157]\) there is a natural isomorphism \((V \otimes V) \otimes V \cong V \otimes (V \otimes V)\) and similarly for higher tensor powers. We therefore write \( V^{\otimes n} \) for the \( n \)-fold tensor product of \( V \) with itself and write elements of \( V^{\otimes n} \) as \( \sum_{i,j_2,\ldots,j_n} c_{ij_2,\ldots,j_n} v_{i_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_n} \) without parentheses. We define \( V^{\otimes 0} \) to be \( k \).

We define the tensor algebra \( T(V) \) to be \( \bigoplus_d V^{\otimes d} \).

(169) Show that \( T(V) \) has a unique ring structure where the product of \( \sigma \in V^{\otimes s} \) and \( \tau \in V^{\otimes t} \) is \( \sigma \otimes \tau \in V^{\otimes (s+t)} \).

(170) Let \( \alpha : V \to W \) be a linear map. Show that there is a unique map of rings \( T(\alpha) : T(V) \to T(W) \) with \( T(\alpha)(v) = \alpha(v) \) for \( v \in V \).

We define the symmetric algebra \( \text{Sym}^*(V) \) to be the quotient of \( T(V) \) by the 2-sided ideal generated by all tensors of the form \( v \otimes w - w \otimes v \).

(171) Show that \( \text{Sym}^*(V) \) is a commutative ring.

(172) Show that \( \text{Sym}^*(V) \) breaks up as a direct sum \( \bigoplus_{d=0}^\infty \text{Sym}^d(V) \) where \( \text{Sym}^d(V) \) is a quotient of \( V^{\otimes d} \).

(173) Let \( x_1, x_2, \ldots, x_n \) be a basis of \( V \). Show that \( \{ x_{i_1},x_{i_2},\ldots,x_{i_d} : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n \} \) is a basis of \( \text{Sym}^d(V) \). Show that \( \text{Sym}^*(V) \cong k[x_1,\ldots,x_n] \).

We define the exterior algebra, \( \wedge^*(V) \) to be the quotient of \( T(V) \) by the two sided ideal generated by \( v \otimes v \) for all \( v \in V \). The multiplication in \( \wedge^*(V) \) is generally denoted \( \wedge \).

(174) Show that, for \( v \) and \( w \in V \), we have \( v \wedge w = -w \wedge v \).

(175) Show that \( \wedge^*(V) \) breaks up as a direct sum \( \bigoplus_{d=0}^\infty \wedge^d(V) \) where \( \wedge^d(V) \) is a quotient of \( V^{\otimes d} \).

(176) Let \( e_1, e_2, \ldots, e_n \) be a basis of \( V \). Show that \( \{ e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} : 1 \leq i_1 < i_2 < \cdots < i_d \leq n \} \) is a basis of \( \wedge^d(V) \).

(177) Let \( v_1, v_2, \ldots, v_d \in V \). Show that \( v_1 \wedge v_2 \wedge \cdots \wedge v_d = 0 \) if and only if \( v_1, v_2, \ldots, v_d \) are linearly dependent.

We now consider the effect of these constructions on linear maps. Let \( V \) and \( W \) be \( k \)-vector spaces and \( \alpha : V \to W \) a linear map.

(178) Show that there are unique ring maps \( \text{Sym}^*(\alpha) : \text{Sym}^*(V) \to \text{Sym}^*(W) \) and \( \wedge^*(\alpha) : \wedge^*(V) \to \wedge^*(W) \) with \( \text{Sym}^*(\alpha)(v) = \alpha(v) \) and \( \wedge^*(\alpha)(v) = \alpha(v) \) for \( v \in V \).

(179) Let \( \alpha : k^3 \to k^3 \) be given by the matrix \( \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} \). Compute the matrix of \( \wedge^2(\alpha) : \wedge^2(k^3) \to \wedge^2(k^3) \).

(180) Let \( \alpha : k^2 \to k^2 \) be given by the matrix \( \begin{bmatrix} p & q \\ r & s \end{bmatrix} \). Compute the matrix of \( \text{Sym}^2(\alpha) : \text{Sym}^2(k^2) \to \text{Sym}^2(k^2) \).

(181) Show that \( \wedge^d(\alpha \circ \beta) = \wedge^d(\alpha) \circ \wedge^d(\beta) \) and \( \text{Sym}^d(\alpha \circ \beta) = \text{Sym}^d(\alpha) \circ \text{Sym}^d(\beta) \).

Given an \( m \times n \) matrix \( X \) with entries in \( k \), and subsets \( I \subseteq \{1,2,\ldots,m\} \) and \( J \subseteq \{1,2,\ldots,n\} \) of the same size, define \( \Delta_{IJ}(X) \) to be the determinant of the square submatrix of \( X \) using rows \( I \) and columns \( J \).

(182) Prove the Cauchy-Binet formula: Let \( X \) and \( Y \) be \( a \times b \) and \( b \times c \) matrices with entries in \( k \) and let \( I \) and \( K \) be subsets of \( \{1,2,\ldots,a\} \) and \( \{1,2,\ldots,c\} \) with \( |I| = |J| = q \). Then

\[
\Delta_{IK}(XY) = \sum_{J \subseteq \{1,2,\ldots,b\} \atop |J| = q} \Delta_{IJ}(X)\Delta_{JK}(Y).
\]