Tensor Products of Modules

Let $R$ be a ring, let $M$ be a right $R$-module and $N$ a left $R$-module. We define the abelian group $M \otimes N$ to be generated by symbols $m \otimes n$, for $m \in M$ and $n \in N$, modulo the relations:

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \ (m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \ (mr) \otimes n = m \otimes (rn).$$

We write $M \otimes_R N$ if the ring $R$ is not clear from context.

(145) Prove the universal property of tensor products: For any abelian group $A$, and any $R$-bilinear pairing $\langle \ , \ \rangle : M \times N \to A$, there is a unique additive map $\lambda : M \otimes N \to A$ such that $\langle m, n \rangle = \lambda(m \otimes n)$. (In the noncommutative setting, $R$-bilinear just means $\langle mr, n \rangle = \langle m, rn \rangle$.)

(146) Let $M_1$ and $M_2$ be right $R$-modules, $N_1$ and $N_2$ be left $R$-modules and $\alpha : M_1 \to M_2$ and $\beta : N_1 \to N_2$ be $R$-linear maps. Show that there is a unique additive map $\alpha \otimes \beta : M_1 \otimes N_1 \to M_2 \otimes N_2$ such that $(\alpha \otimes \beta)(m \otimes n) = \alpha(m) \otimes \beta(n)$.

(147) Let $M_1$, $M_2$, $M_3$ be right $R$-modules, let $N_1$, $N_2$ and $N_3$ be left $R$-modules, and let $\alpha_1 : M_1 \to M_2$, $\alpha_2 : M_2 \to M_3$, $\beta_1 : N_1 \to N_2$ and $\beta_2 : N_2 \to N_3$ be $R$-linear maps. Show that $(\alpha_2 \circ \beta_2) \circ (\alpha_1 \otimes \beta_1) = (\alpha_2 \circ \alpha_1) \otimes (\beta_2 \otimes \beta_1)$.

Tensor products over noncommutative rings are important, but we will mostly focus on the commutative case.

From now on, let $R$ be a commutative ring.

(148) Let $M$ and $N$ be $R$-modules. Show that there is a unique $R$-module structure on $M \otimes N$ such that $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$.

(149) Let $M$, $N$ and $A$ be $R$-modules and let $M \otimes N \to A$ be an $R$-bilinear pairing (which now means that $\langle rm, n \rangle = \langle m, rn \rangle = r\langle m, n \rangle$). Show that the unique map $\lambda : M \otimes N \to A$ coming from the universal property of tensor products is $R$-linear.

(150) Suppose that $I$ is a subset of $M$ which generates $M$ as an $R$-module, and $J$ is a subset of $N$ which generates $N$ as an $R$-module. Show that the elements $m \otimes n$, $m \in I$ and $n \in J$, generate $M \otimes N$ as an $R$-module.

(151) Show that $(M_1 \oplus M_2) \otimes_R N \cong M_1 \otimes_R N \oplus M_2 \otimes_R N$ and $M \otimes_R (N_1 \oplus N_2) \cong M \otimes_R N_1 \oplus M \otimes_R N_2$.

(152) Show that $R \otimes_R M \cong M$ and $M \otimes_R R \cong M$.

(153) Show that $R^{\otimes m} \otimes R^{\otimes n} \cong R^{\otimes mn}$.

So far we have emphasized the similarities between tensor products of vector spaces and of modules, but there are important differences.

(154) It is not true that, if $M_1 \to M_2$ is injective then $M_1 \otimes N \to M_2 \otimes N$ is injective. Give a counterexample. One of the easiest takes $R = \mathbb{Z}$ and $N = \mathbb{Z}/2\mathbb{Z}$.

(155) It is not true that, if $m \in M$ and $n \in N$ are nonzero, then $m \otimes n$ is nonzero in $M \otimes N$. Give a counterexample. One of the easiest takes $R = \mathbb{Z}$. 