13. ABELIAN EXTENSIONS

Here is a lemma from the homework; check that everyone in your group solved it.

**Problem 13.1.** Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be short exact. Let $\tilde{C}$ be any subset of $G$ such that $\beta : \tilde{C} \to C$ is bijective. Then every $b \in B$ can be uniquely written in the form $\alpha(a)\tilde{c}$ for $a \in A$ and $\tilde{c} \in \tilde{C}$.

In this worksheet, we will study short exact sequences $1 \to A \to G \to H \to 1$ with $A$ abelian; when such a short exact sequence exists, we say that $G$ is an **abelian extension** of $H$. A special case is when $A$ is central in $G$, in this case, we say that $G$ is a **central extension** of $H$.

**Problem 13.2.** Let $1 \to A \to G \to H \to 1$ be an abelian extension. Since $A$ is normal in $G$, we get an action of $G$ on $A$ by $g : a \mapsto gag^{-1}$. Show that the map $G \to \text{Aut}(A)$ factors through $H$.

We’ll write $\phi : H \to \text{Aut}(A)$ for the resulting action.

**Problem 13.3.** Show that the action $\phi$ is trivial (meaning $\phi(h)(a) = a$ for all $h \in H$ and $a \in A$) if and only if the extension $1 \to A \to G \to H \to 1$ is central.

Classifying abelian extensions with fixed $(A, H)$ thus comes down to two parts (1) classifying all actions of $H$ on $A$ and (2) for each action $\phi$, classifying all abelian extensions that result. We know there is always at least one such extension: the semidirect product $A \rtimes H$.

**Problem 13.4.** Let $p$ be a prime number, let $H$ be a group of order $p^k$ and let $1 \to C_p \to G \to H$ be an abelian extension. Show that it must be a central extension.

**Problem 13.5.** Let $n$ be a positive integer and let $1 \to Z \to G \to C_n \to 1$ be a central extension. Show that $G$ is abelian. (Hint: Let $g \in G$ map to a generator of $C_n$. Use Problem 13.1 with $S = \{1, g, g^2, \ldots, g^{n-1}\}$.)

**Problem 13.6.** Let $p$ be a prime number and let $G$ be a group of order $p^k$. Show that $G$ lies in a central extension $1 \to C_p \to G \to H \to 1$ for some $H$ of order $p^{k-1}$.

**Problem 13.7.** Let $p$ be prime. Show that every group of order $p^2$ is isomorphic to $C_p^2$ or $C_p \times C_p$.

**Problem 13.8.** Let $p$ and $q$ be distinct prime numbers, let $A \cong C_p$, $H \cong C_q$ and let $1 \to A \to G \to H \to 1$ be an abelian extension.

1. If $p \not\equiv 1 \mod q$, show that the action of $H$ on $A$ is trivial.
2. If the action of $H$ on $A$ is trivial, show that $G \cong C_{pq} \cong C_p \times C_q$.
3. If the action $\phi$ of $H$ on $A$ is nontrivial, show that $G \cong C_p \rtimes_{\phi} C_q$.

**Problem 13.9.** Let $p$ be an odd prime, let $A \cong C_p$, $H \cong C_p^2$. In this problem, we will classify abelian extensions $1 \to A \to G \to H \to 1$. We write $z$ for a generator of $A$ and $x$ and $\tilde{y}$ for lifts of $x$ and $y$ to $G$.

1. Show that $z$ is central in $G$. (Hint: What can $\phi$ be?)
2. Show that every element of $G$ is uniquely of the form $\tilde{x}^a\tilde{y}^b\tilde{z}^c$ for $a, b, c \in \{0, 1, \ldots, p-1\}$.
3. Show that $\tilde{x}^p$, $\tilde{y}^p$ and $\tilde{y}^i\tilde{x}^{-1}\tilde{y}^{-1}$ are of the form $z^k$, $z^j$ and $z^h$ for some $i, j$ and $k \in \mathbb{Z}/p\mathbb{Z}$.
4. Suppose that $k \equiv 0 \mod p$. Show that $G$ is abelian and is isomorphic to either $C_p^3$ or $C_{p^2} \times C_p$.
5. Suppose that $k \not\equiv 0 \mod p$ and $(i, j) = (0, 0)$. Show that $(\tilde{x}^{a_1}\tilde{y}^{b_1}\tilde{z}^{c_1})(\tilde{x}^{a_2}\tilde{y}^{b_2}\tilde{z}^{c_2}) = \tilde{x}^{a_1+a_2}\tilde{y}^{b_1+b_2}\tilde{z}^{c_1+c_2+kb_1a_2}$.

Show that $G$ is isomorphic to the group of matrices of the form $\begin{bmatrix} 1 & * \\ * & 1 \end{bmatrix}$ with entries in $\mathbb{Z}/p\mathbb{Z}$.

6. Suppose that $(i, j) \neq (0, 0)$. Show that there are $a$ and $b$ not both $0 \mod p$ such that $(\tilde{x}^a\tilde{y}^b)^p = 1$.

This is where you will need that $p$ is odd.

7. Suppose that $(i, j) \neq (0, 0)$ Show that $G \cong C_{p^2} \rtimes_{\phi} C_p$ and describe the action of $C_p$ on $C_{p^2}$.

**Problem 13.10.** Let $p$ be an odd prime. Show that every group of order $p^3$ is isomorphic to one of $C_p^3$, $C_{p^2} \times C_p$, $C_{p^2} \times C_p$, $C_{p^2} \rtimes C_p$, $\left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\}$.