17. Degrees of Field Extensions, and Constructible Numbers

**Definition:** Let $L$ be a field and $K$ a subfield. The **degree of $L$ over $K$**, written $[L : K]$, is the dimension of $L$ as a $K$-vector space.

**Problem 17.1.** Let $K \subseteq L \subseteq M$ be three fields with $[L : K]$ and $[M : L] < \infty$. Show that $[M : K] = [M : L][L : K]$.

**Problem 17.2.** Let $k \subseteq K$ be a field extension with $[K : k] < \infty$. Let $\theta \in K$ and let $m(x)$ be the minimal polynomial of $\theta$ over $k$. Show that $\deg m(x)$ divides $[K : k]$.

We illustrate these results with an extremely classical application. A real number $\theta \in \mathbb{R}$ is called **constructible** if it can be written in terms of rational numbers using the operations $+, -, \times, \div$ and $\sqrt{\cdot}$. Classically, these numbers were studied because the distance between any two points constructed with straightedge and compass is constructible; now we can motivate them by saying they are the numbers which can be computed exactly with a four function calculator.

![Two ancient mathematical tools](image)

**Figure:** Two ancient mathematical tools

**Problem 17.3.** Suppose we compute a sequence of real numbers $\theta_1, \theta_2, \theta_3, \ldots, \theta_N$ where each $\theta_k$ is either

- a rational number,
- of one of the forms $\theta_i + \theta_j, \theta_i - \theta_j, \theta_i \theta_j$ or $\theta_i / \theta_j$ for some $i, j < k$ or
- of the form $\sqrt{\theta_j}$ for some $j < k$.

Show that $[\mathbb{Q}[\theta_1, \theta_2, \ldots, \theta_N] : \mathbb{Q}]$ is a power of 2.

**Problem 17.4.** Let $\theta$ be a constructible real number and let $m(x)$ be its minimal polynomial over $\mathbb{Q}$. Show that $\deg m(x)$ is a power of 2.

**Problem 17.5.** *(The impossibility of doubling the cube).* Show that $\sqrt[3]{2}$ is not constructible.

**Problem 17.6.** *(The impossibility of trisecting the angle)* It is well known that a $60^\circ$ angle is constructible with straightedge and compass. Show, however, that $\cos 20^\circ$ is not constructible. Hint:

$$4 \cos^3 20^\circ - 3 \cos 20^\circ = \cos 60^\circ = \frac{1}{2}.$$