18. Splitting fields and maps between them

**Definition:** Let $k$ be a field, let $f(x)$ be a polynomial in $k[x]$ and let $K$ be an extension field of $f$. We will say that $f$ **splits in $K$** if $f$ factors as a product of linear polynomials in $K[x]$. We say that $K$ is a **splitting field of $f$** if $f$ splits as a product $c \prod (x - \theta_j)$ in $K[x]$ and the field $K$ is generated by $k$ and by the $\theta_j$.

For example, if $k = \mathbb{Q}$ and $\theta_1, \theta_2, \ldots, \theta_n$ are the roots of $f(x)$ in $\mathbb{C}$, then $\mathbb{Q}[\theta_1, \ldots, \theta_n]$ is a splitting field of $f(x)$.

**Problem 18.1.** Let $k$ be a field and let $f(x)$ be a polynomial in $k[x]$. Show that $f$ has a splitting field.

**Problem 18.2.** Let $L$ be a splitting field for $x^3 - 2$ over $\mathbb{Q}$. Show that $[L : \mathbb{Q}] = 6$. (Hint: At one point, it will be very useful to use the fact that $\mathbb{Q}[\sqrt[3]{2}]$ is a subfield of $\mathbb{R}$.)

**Problem 18.3.** Let $L = \mathbb{C}(x_1, x_2, \ldots, x_n)$. Let $e_k$ be the $k$-th elementary symmetric polynomial and let $K = \mathbb{C}(e_1, e_2, \ldots, e_n) \subset L$. Show that $L$ is a splitting field for $x^n - e_1x^{n-1} + e_2x^{n-2} - \cdots \pm e_n$ over $K$.

**Problem 18.4.** Let

$$f(x) = \left(x - \cos \frac{2\pi}{7}\right) \left(x - \cos \frac{4\pi}{7}\right) \left(x - \cos \frac{8\pi}{7}\right) = \frac{1}{8} \left(8x^3 + 4x^2 - 4x - 1\right).$$

I promise, and you may trust me, that $f(x)$ is irreducible. Let $K = \mathbb{Q}(\cos \frac{2\pi}{7})$.

1. Show that $[K : \mathbb{Q}] = 3$.
2. Show that $f(x)$ splits in $K$. Hint: Use the double angle formula.
3. Show that there is an automorphism $\sigma : K \to K$ with $\sigma(\cos \frac{2\pi}{7}) = \cos \frac{4\pi}{7}$.

**Problem 18.5.** Let $k$ be a field and let $f(x)$ be a polynomial in $k[x]$. Let $K$ be a splitting field of $f$ in which $f$ splits as $\prod (x - \alpha_j)$. Let $\sigma : k \to L$ be a field homomorphism and let $\sigma(f)(x) := \sum \sigma(f_j)x^j$ split in $L$. Show that there is an injection $\phi : K \to L$ making the diagram

```
                 k
                 ↓ \sigma
                 ↓   \phi
             K -----+----- L
                 ↓   \phi
            L
```

commute. Hint: Think about $k \subseteq k[\alpha_1] \subseteq k[\alpha_1, \alpha_2] \subseteq \cdots \subseteq k[\alpha_1, \alpha_2, \ldots, \alpha_n] = K$.

**Problem 18.6.** Let $k_1$ and $k_2$ be two fields and let $\sigma : k_1 \to k_2$ be an isomorphism. Let $(x) = \sum f_jx^j$ be a polynomial in $k_1[x]$ and let $\sigma(f)(x) := \sum \sigma(f_j)x^j$. Let $K_1$ be a splitting field of $f$ and let $K_2$ be a splitting field of $\sigma(f)$. Show that there is an isomorphism $K_1 \cong K_2$ making the diagram

```
                 k_1  \sigma  \to  k_2
                 ↓       \cong
             K_1 -----+----- K_2
                 ↓   \cong
            L
```

commute.

**Problem 18.7.** Let $k$ be a field and let $f(x)$ be a polynomial in $k[x]$. Let $K_1$ and $K_2$ be two splitting fields of $f$. Show that there is an isomorphism $K_1 \cong K_2$ making the diagram

```
                k
           ↓    \cong
           ↓   \cong
       K_1 -----+----- K_2
```

commute. So splitting fields are unique.