18. DEGREES OF FIELD EXTENSIONS, AND CONSTRUCTIBLE NUMBERS

**Definition:** Let \( L \) be a field and \( K \) a subfield. The **degree of \( L \) over \( K \)**, written \([L : K]\), is the dimension of \( L \) as a \( K \)-vector space.

**Problem 18.1.** Let \( K \subseteq L \subseteq M \) be three fields with \([L : K]\) and \([M : L] < \infty\). Show that \([M : K] = [M : L][L : K]\).

**Problem 18.2.** Let \( k \subseteq K \) be a field extension with \([K : k] < \infty\). Let \( \theta \in K \) and let \( m(x) \) be the minimal polynomial of \( \theta \) over \( k \). Show that \( \deg m(x) \) divides \([K : k]\).

We illustrate these results with an extremely classical application. A real number \( \theta \in \mathbb{R} \) is called **constructible** if it can be written in terms of rational numbers using the operations \(+, -, \times, \div\) and \( \sqrt{\cdot} \). Classically, these numbers were studied because the distance between any two points constructed with straightedge and compass is constructible; now we can motivate them by saying they are the numbers which can be computed exactly with a four function calculator.

![Figure: Two ancient mathematical tools](image)

**Problem 18.3.** Suppose we compute a sequence of real numbers \( \theta_1, \theta_2, \theta_3, \ldots, \theta_N \) where each \( \theta_k \) is either
- a rational number,
- of one of the forms \( \theta_i + \theta_j, \theta_i - \theta_j, \theta_i \theta_j \) or \( \theta_i/\theta_j \) for some \( i, j < k \) or
- of the form \( \sqrt{\theta_j} \) for some \( j < k \).

Show that \([\mathbb{Q}[\theta_1, \theta_2, \ldots, \theta_N] : \mathbb{Q}]\) is a power of 2.

**Problem 18.4.** Let \( \theta \) be a constructible real number and let \( m(x) \) be its minimal polynomial over \( \mathbb{Q} \). Show that \( \deg m(x) \) is a power of 2.

**Problem 18.5. (The impossibility of doubling the cube.)** Show that \( \sqrt[3]{2} \) is not constructible.

**Problem 18.6. (The impossibility of trisecting the angle)** It is well known that a 60° angle is constructible with straightedge and compass. Show, however, that \( \cos 20^\circ \) is not constructible. Hint:

\[
4 \cos^3 20^\circ - 3 \cos 20^\circ = \cos 60^\circ = \frac{1}{2}.
\]