**Definition:** Let $K \subseteq L$ be fields. An *automorphism* of $L$ is a bijection $\sigma : L \to L$ with $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$. An *automorphism of $L$ fixing $K$* is an automorphism of $L$ obeying $\sigma(a) = a$ for all $a \in K$. We write $\text{Aut}(L)$ for the automorphisms of $L$ and $\text{Aut}(L/K)$ for the automorphisms of $L$ fixing $K$.

**Problem 19.1.** Let $K \subseteq L$ be fields. Let $f(x)$ be a polynomial in $K[x]$; let $\{\theta_1, \theta_2, \ldots, \theta_r\}$ be the roots of $f$ in $L$.

1. Show that $\text{Aut}(L/K)$ maps $\{\theta_1, \theta_2, \ldots, \theta_r\}$ to itself.
2. Show that stabilizer of $\theta_j$ in $\text{Aut}(L/K)$ is $\text{Aut}(L/K(\theta_j))$.
3. Let $L = \mathbb{Q}(\sqrt[4]{2})$ and let $f(x) = x^2 - 2$. Show that the roots of $f(x)$ in $L$ are $\{\pm \sqrt{2}\}$ and show that $\text{Aut}(L/\mathbb{Q})$ fixes both of them.

**Problem 19.2.** Let $K$ be a field, let $f$ be a polynomial in $K[x]$, let $L$ be a splitting field for $f$ and let $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be the roots of $f$ in $L$. Assume $\{\theta_1, \theta_2, \ldots, \theta_n\}$ are distinct.

1. Show that the action of $\text{Aut}(L/K)$ takes $\{\theta_1, \theta_2, \ldots, \theta_n\}$ to itself.
2. Show that this action of $\text{Aut}(L/K)$ gives an injection $\text{Aut}(L/K) \hookrightarrow S_n$.

**Problem 19.3.** Let $K, f, L$ and $\{\theta_1, \theta_2, \ldots, \theta_m\}$ be as in Problem 19.2. Let $g(x)$ be an irreducible factor of $f(x)$ in $K[x]$ and renumber the $\theta$’s so that $\{\theta_1, \theta_2, \ldots, \theta_m\}$ are the roots of $g$ in $L$. Show that $\{\theta_1, \theta_2, \ldots, \theta_m\}$ is the $\text{Aut}(L/K)$-orbit of $\theta_1$ in $L$. Hint: Apply Problem 18.6 to the diagram

$$
\begin{array}{c}
K[\theta_i] \preceq K[x]/g(x)K[x] \cong K[\theta_j] \\
\downarrow \\
L \hookrightarrow \cdots \to L
\end{array}
$$

**Problem 19.4.** Let $L$ be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Show that $\text{Aut}(L/\mathbb{Q}) \cong S_3$.

**Problem 19.5.** Let $L = \mathbb{Q}(\cos \frac{2\pi}{7})$. Show that $\text{Aut}(L/\mathbb{Q}) \cong C_3$.

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1This happens if and only if $\text{GCD}(f(x), f'(x)) = 1$, see the homework.