I added up the area of my two squares: 1300. The side of one exceeds the side of the other by 10.

Babylonian tablet, 2000-1600 BCE, British Museum

Problem 1.1. Let \( x^2 + bx + c \) be a polynomial with complex coefficients and let its roots be \( \alpha_1 \) and \( \alpha_2 \). Express the following quantities in terms of \( \alpha_1 \) and \( \alpha_2 \). In the expressions with a square root, you may choose which square root to use.

\[
\begin{align*}
    b & \quad c \\
    -b + \sqrt{b^2 - 4c} & \quad -b - \sqrt{b^2 - 4c} \\
\end{align*}
\]

Problem 1.2. Let the symmetric group \( S_2 \) act by switching \( \alpha_1 \) and \( \alpha_2 \). Describe the effect of \( S_2 \) on each of the expressions you derived in Problem 1.1.

When the cube with the case beside it / equates itself to some other whole number …

Tartaglia, 1543

Let \( \omega = -\frac{1 + \sqrt{-3}}{2} \); we recall that \( \omega^2 + \omega + 1 = 0 \) and \( \omega^3 = 1 \). Let \( \beta_1, \beta_2 \) and \( \beta_3 \) be complex numbers. Define:

\[
\begin{align*}
    s & = \beta_1 + \beta_2 + \beta_3 \\
    \sigma_1 & = \beta_1 + \omega \beta_2 + \omega^2 \beta_3 \\
    \sigma_2 & = \beta_1 + \omega^2 \beta_2 + \omega \beta_3 \\
\end{align*}
\]

Problem 1.3. Let \( S_3 \) permute \( \beta_1, \beta_2, \beta_3 \).

(1) Describe how \( S_3 \) acts on \( \{\omega \sigma_1, \omega^2 \sigma_1, \sigma_2, \omega \sigma_2, \omega^2 \sigma_2\} \).

(2) Describe how \( S_3 \) acts on \( \{\sigma_3, \sigma_3^2\} \).

(3) Show that \( S_3 \) fixes \( s \) and the coefficients of the quadratic polynomial \( y^2 - f_1 y + f_2 := (y - \sigma_1^2)(y - \sigma_3^2) \).

Let \( (x - \beta_1)(x - \beta_2)(x - \beta_3) = x^3 - e_1 x^2 + e_2 x - e_3 \). To make your lives easier, here are some useful formulas:

\[
\begin{align*}
    e_1 & = \beta_1 + \beta_2 + \beta_3 \\
    e_2 & = \beta_1 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_3 \\
    e_3 & = \beta_1 \beta_2 \beta_3 \\
    e_1^2 & = \beta_1^2 + 2\beta_1 \beta_2 + 2\beta_1 \beta_3 + \beta_2^2 + 2\beta_2 \beta_3 + \beta_3^2 \\
    e_3^2 & = \beta_1^3 + 3\beta_1^2 \beta_2 + 3\beta_1 \beta_2^2 + 6\beta_1 \beta_2 \beta_3 + 3\beta_1 \beta_3^2 + \beta_2^3 + 3\beta_2^2 \beta_3 + 3\beta_2 \beta_3^2 + \beta_3^3 \\
    \sigma_1 \sigma_2 & = \beta_1^2 - \beta_1 \beta_2 - \beta_1 \beta_3 + \beta_2^2 - \beta_2 \beta_3 + \beta_3^2 \\
    \sigma_1^2 + \sigma_2^2 & = 2\beta_1^3 - 3\beta_1^2 \beta_2 - 3\beta_1 \beta_2^2 - 3\beta_1 \beta_2 \beta_3 + 12\beta_1 \beta_2 \beta_3 - 3\beta_1 \beta_3^2 + 2\beta_2^3 - 3\beta_2 \beta_3^2 + 3\beta_2 \beta_3^2 + 2\beta_3^3 \\
\end{align*}
\]

Problem 1.4. Give formulas for the following, as polynomials in \( e_1, e_2, e_3 \):

\[
\begin{align*}
    s & \quad \sigma_1 \sigma_2 \\
    f_1 & \quad f_2 \\
\end{align*}
\]

Problem 1.5. Show that \( \sigma_1 \) and \( \sigma_2 \) can be computed from \( e_1, e_2, e_3 \) using the operations \( +, -, \times, \sqrt{\cdot} \) and \( \sqrt{\cdot} \), together with multiplication by rational numbers and the number \( \omega \). Show how to likewise compute \( \beta_1, \beta_2, \beta_3 \).

Given an equation in which the unknown quantity has four dimensions … reduce it to another of the third degree, in the following manner …

Descartes, La Géométrique, 1637

Let \( \gamma_1, \gamma_2, \gamma_3 \) \( \gamma_4 \) be complex numbers. Set

\[
\begin{align*}
    t & = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \\
    \tau_1 & = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4 \\
    \tau_2 & = \gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 \\
    \tau_3 & = \gamma_1 - \gamma_2 - \gamma_3 + \gamma_4 \\
\end{align*}
\]

Problem 1.6. Let \( S_4 \) permute \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \). Describe how \( S_4 \) acts on

(1) \( \{\pm \tau_1, \pm \tau_2, \pm \tau_3\} \)

(2) \( \{\tau_1^2, \tau_2^2, \tau_3^2\} \)

(3) Show that \( S_4 \) fixes \( t \) and the coefficients of the polynomial \( (x - \tau_1^2)(x - \tau_2^2)(x - \tau_3^2) \).

Problem 1.7. How would you compute the \( \gamma_i \) from the coefficients of the quartic \( \prod (x - \gamma_i) \), using the operations \( +, -, \times, \div \) and \( \sqrt{\cdot} \)?