19. Splitting fields

**Definition:** Let \( k \) be a field, let \( f(x) \) be a polynomial in \( k[x] \) and let \( K \) be an extension field of \( f \). We will say that \( f \) splits in \( K \) if \( f \) factors as a product of linear polynomials in \( K[x] \). We say that \( K \) is a splitting field of \( f \) if \( f \) splits as a product \( c \prod (x - \theta_j) \) in \( K[x] \) and the field \( K \) is generated by \( k \) and by the \( \theta_j \).

For example, if \( k = \mathbb{Q} \) and \( \theta_1, \theta_2, \ldots, \theta_n \) are the roots of \( f(x) \) in \( \mathbb{C} \), then \( \mathbb{Q}[\theta_1, \ldots, \theta_n] \) is a splitting field of \( f(x) \).

**Problem 19.1.** Let \( k \) be a field and let \( f(x) \) be a polynomial in \( k[x] \). Show that \( f \) has a splitting field. (Please do not use that every field has an algebraic closure. That is a much harder result than this one.)

**Problem 19.2.** Let
\[
f(x) = (x - \cos \frac{2\pi}{7})(x - \cos \frac{4\pi}{7})(x - \cos \frac{8\pi}{7}) = \frac{1}{8}(8x^3 + 4x^2 - 4x - 1).
\]
I promise, and you may trust me, that \( f(x) \) is irreducible.\(^1\) Let \( K = \mathbb{Q}(\cos \frac{2\pi}{7}) \).

1. Show that \([K : \mathbb{Q}] = 3\).
2. Show that \( f(x) \) splits in \( K \). Hint: Use the double angle formula.
3. Show that there is an automorphism \( \sigma : K \to K \) with \( \sigma(\cos \frac{2\pi}{7}) = \cos \frac{4\pi}{7} \).

**Problem 19.3.** Let \( L \) be a splitting field for \( x^3 - 2 \) over \( \mathbb{Q} \). Show that \([L : \mathbb{Q}] = 6\). (Hint: At one point, it will be very useful to use the fact that \( \mathbb{Q}[\sqrt[3]{2}] \) is a subfield of \( \mathbb{R} \).)

This is a good time to discuss separable polynomials.

**Definition:** Let \( k \) be a field and let \( f(x) \) be a polynomial in \( k[x] \). We say \( f \) is separable if \( \text{GCD}(f(x), f'(x)) = 1 \).

**Problem 19.4.** Let \( k \) be a field, let \( f(x) \) be a polynomial in \( k[x] \) and let \( K \) be a field where \( f \) splits as \( c \prod_{j=1}^{n} (x - \theta_j) \). Show that \( f \) is separable if and only if \( \theta_1, \theta_2, \ldots, \theta_n \) are distinct.

**Problem 19.5.** Let \( k \) be a field of characteristic zero.

1. Show that a polynomial in \( k[x] \) is separable if and only if it is square free.
2. Show that irreducible polynomials in \( k[x] \) are separable.

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\(^1\)In particular, the fact that every field embeds in an algebraically closed field uses the Axiom of Choice, and this problem does not.

\(^2\)The most straightforward way to check this is to use the rational root theorem. The slickest is to note that \( f(x + 1) = \frac{1}{8}(8x^3 + 28x^2 + 28x + 7) \) and apply Eisenstein’s irreducibility theorem.