Problem 2.1. Let $p$ be a prime number and let $G$ be a group of order $p$. Show that $G$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Problem 2.2. Let $V$ be a vector space over some field. Let $G$ be the set $V \times \wedge^2 V$ and define a multiplication operation on $G$ by $(v, \alpha) \ast (w, \beta) = (v + w, \alpha + \beta + v \wedge w)$. Check that $G$ is a group.

Problem 2.3. Let $G$ be a group with $n$ elements.

1. Show that $G$ is isomorphic to a subgroup of $S_n$.
2. Let $k$ be a field. Show that $G$ is isomorphic to a subgroup of $GL_n(k)$.

Problem 2.4. Let $G$ be a group and let $g \in G$. The conjugacy class of $g$, written $\text{Conj}(g)$, is $\{hgh^{-1} : h \in G\}$ and the centralizer of $g$, written $Z(g)$, is $\{h \in G : gh = hg\}$. Suppose that $G$ is finite and let $g \in G$. Show that
\[
\#(G) = \#\text{Conj}(g) \cdot \#Z(g).
\]

Problem 2.5. Let $p$ be a prime and let $G$ be a group of order $p^k$ for some $k \geq 1$.

1. Show that there is some $g \in G$ other than $e$ such that $\text{Conj}(g) = \{g\}$.
2. Show that this $g$ commutes with every $h \in G$.

Problem 2.6. Let $G$ be a group and let $H$ be a subgroup of $G$ with $[G : H] = n$.

1. Show that there is a normal subgroup $N$ of $G$ with $N \leq H \leq G$ and $[G : N]$ dividing $n!$.
2. Suppose that $n$ is prime and that $|G|$ is not divisible by any prime $< n$. Show that $H$ is normal in $G$.

Problem 2.7. In this problem, we will check that $A_n$ is simple for $n > 5$. You may assume that we already know $A_5$ is simple. Let $N$ be a normal subgroup of $A_n$ other than $\{e\}$. We will show that $N = A_n$.

Let $\sigma \in N$ with $\sigma \neq e$. Choose $i$ with $\sigma(i) \neq i$ and choose $j \not\in \{\sigma^{-1}(i), i, \sigma(i)\}$. Put $\tau = (i \sigma(i) j)$. Put $\gamma = \sigma^{-1} \tau^{-1} \sigma \tau$. Let $X$ be a five element subset of $\{1, 2, \ldots, n\}$ containing $\{\sigma^{-1}(i), i, \sigma(i), j, \sigma^{-1}(j)\}$. Let $A_X$ be those permutations in $A_n$ which fix all $x \not\in X$.

1. Show that $\gamma \in N$, that $\gamma \neq e$ and that $\gamma \in A_X$.
2. Show that $N \cong A_X$. Hint: Notice that $N \cap A_X$ is normal in $A_X$.
3. Show that $N$ contains a permutation of the form $(xyz)$.
4. Show that $N = A_n$.

Problem 2.8. Let $R$ be a ring. An $R$-module $M$ is called simple if $M$ has exactly two submodules, $\{0\}$ and $M$. Note that the zero module is not considered simple. The three parts of this problem are logically independent and can be done in any order, but share many techniques.

1. (Schur’s lemma) Let $S$ be a simple $R$-module and let $\phi : S \to S$ be a nonzero $R$-linear homomorphism. Show that $\phi$ is invertible.
2. Let $S_1$ and $S_2$ be two simple $R$-modules and let $M$ be an $R$-submodule of $S_1 \oplus S_2$. Show that one of the following must hold: (a) $M = 0$ (b) $M = S_1 \oplus 0$ (c) $M = 0 \oplus S_2$ (d) $M = S_1 \oplus S_2$. or (e) $S_1 \cong S_2$ and $M = \{x, \phi(x) : x \in S_1\}$ for some isomorphism $\phi : S_1 \to S_2$.
3. Let $M$ be an $R$-module and let $N_1$ and $N_2$ be $R$-submodules such that $M/N_1$ and $M/N_2$ are simple. Show that either $N_1 = N_2$ or $M/(N_1 \cap N_2) \cong M/N_1 \oplus M/N_2$.

\[\text{If } \sigma^{-1}(i), i, \sigma(i), j \text{ and } \sigma^{-1}(j) \text{ are not distinct, add some extra elements to make } \#(X) = 5.\]