Problem 3.1. Describe all actions of $C_2$ (by group homomorphisms) on the group $C_{21}$.

Problem 3.2. Let $G$ be a group and let $g$ be an element of $G$. Define an action of $\mathbb{Z}$ on $G$ by $\phi(k) = g^k h g^{-k}$. Show that $G \times \mathbb{Z} \simeq G \times \mathbb{Z}$. This gives an example of how two different actions can give isomorphic semi-direct products. (Problem cancelled because we won’t get to semidirect products soon enough.)

Problem 3.3. Does there exist a group $G$ with normal subgroups $N_1$ and $N_2$ such that $N_1 \cong S_5$, $N_2 \cong S_7$, $G/N_1 \cong S_{42}$ and $G/N_2 \cong S_{41}$?

Problem 3.4. Let $G$ be a group and $N$ a normal subgroup. Let $g \in N$ and recall that $Z_G(g) = \{h \in G : gh = hg\}$.

1. Let $h \in G$, so $g$ and $hgh^{-1}$ are both in $N$ and are conjugate in $G$. Show that $hgh^{-1}$ is conjugate to $g$ in $N$ if and only if $h \in Z_G(g) N$.
2. Show that the $S_5$-conjugacy class of $(123)$ is also an $A_5$-conjugacy class, but that the $S_5$-conjugacy class of $(12345)$ splits into two $A_5$-conjugacy classes.
3. For each of the following elements of $SL_2(\mathbb{R})$, describe how the $GL_2(\mathbb{R})$ conjugacy class splits into conjugacy classes of $SL_2(\mathbb{R})$:

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$  

Problem 3.5. Let $G$ be a group and $H$ a subgroup. The subgroup $H$ is called canonical if $\phi(H) = H$ for every automorphism $\phi$ of $G$.

1. Show that, if $H$ is canonical in $G$, then $H$ is normal in $G$.

Let $A$ be a subgroup of $B$, which is a subgroup of $C$. For each of these statements, give a proof or counterexample:

2. If $A$ is canonical in $B$ and $B$ is canonical in $C$, then $A$ is canonical in $C$.
3. If $A$ is canonical in $B$ and $B$ is normal in $C$, then $A$ is normal in $C$.
4. If $A$ is normal in $B$ and $B$ is canonical in $C$, then $A$ is normal in $C$.
5. If $A$ is normal in $B$ and $B$ is normal in $C$, then $A$ is normal in $C$.

Problem 3.6. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence of groups.

1. A left splitting of this sequence is a map $t : B \to A$ with $\lambda \circ \alpha = \text{Id}_A$. Show that, if the sequence $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ has a left splitting, then $B \cong A \times C$.
2. A right splitting of this sequence is a map $r : C \to B$ such that $\beta \circ r = \text{Id}_C$. Show that, if the sequence $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ has a right splitting, then $B \cong A \times C$ for an action of $A$ on $C$. Hint: Apply Worksheet Problem 8.6 to the subgroups $\alpha(A)$ and $\rho(C)$ of $B$. (Problem cancelled because we won’t get to semidirect products soon enough.)

Problem 3.7. Let $R$ be a ring and let $M$ be an $R$-module. A Jordan-Holder filtration of $M$ is a chain of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ such that $M_k/M_{k-1}$ is simple for $1 \leq k \leq \ell$. A module which has a Jordan-Holder filtration is said to have finite length.

1. This problem was on the 593 problem sets, but please write it out to check that you remember it: Let $M$ have finite length and let $N$ be a submodule of $M$. Show that $N$ and $M/N$ have finite length.

In this problem, we will prove the Jordan-Holder Theorem for modules which states: Let $0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ and $0 = M'_0 \subset M'_1 \subset \cdots \subset M'_\ell' = M$ be two Jordan-Holder filtrations. Then $\ell = \ell'$ and there is a permutation $\sigma$ of $\{1, 2, \ldots, \ell\}$ such that $M_k/M_{k-1} \cong M'_{\sigma(k)}/M'_{\sigma(k)-1}$. Our proof is by induction on $\min(\ell, \ell')$.

2. Do the base case $\min(\ell, \ell') = 1$.
3. Explain why we are done if $M_{\ell-1} = M'_{\ell'-1}$.
4. Suppose that $M_{\ell-1} \neq M'_{\ell'-1}$ and put $N = M_{\ell-1} \cap M'_{\ell'-1}$. Explain why we are also done in this case.

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1. When I google “left splitting”, Google’s first suggestion is “left splitting headache”. I hope this will not be your opinion!
2. Look back to Problem 2.8 for previous results on simple modules. You may use any of the parts of that problem in this one.