3. Groups

**Definition.** A group $G$ is a set with a binary operation $*: G \times G \to G$ obeying the properties

1. There is an element 1 of $G$ such that $1 * g = g * 1 = g$ for all $g \in G$.
2. For all $g \in G$, there is an element $g^{-1}$ obeying $g * g^{-1} = g^{-1} * g = 1$.
3. For all $g_1, g_2, g_3 \in G$, we have $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.

Given a group $G$, a subgroup of $G$ is a subset containing 1 and closed under $*$ and $g \to g^{-1}$.

Depending on context, we may denote $*$ by $\times$, $\cdot$ or no symbol at all, and we may denote 1 as 1, $e$ or $\text{Id}$.

**Problem 3.1.** Show that a group $G$ only has one element 1 obeying the condition (1).

**Problem 3.2.** Let $G$ be a group and let $g \in G$. Show that $G$ only has one element obeying the condition (2).

**Definition.** Given two groups $G$ and $H$, a group homomorphism is a map $\phi : G \to H$ obeying $\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$. A bijective group homomorphism is called an isomorphism and two groups are called isomorphic if there is an isomorphism between them.

A group homomorphism can also be called a “map of groups” or a “group map”.

**Problem 3.3.** Let $\phi : G \to H$ be a group homomorphism. Show that $\phi(1) = 1$ and $\phi(g^{-1}) = \phi(g)^{-1}$.

**Problem 3.4.** Let $\phi : G \to H$ be a group homomorphism.

1. The image of $\phi$ is $\text{Im}(\phi) := \{\phi(g) : g \in G\}$. Show that $\text{Im}(\phi)$ is a subgroup of $G$.
2. The kernel of $\phi$ is $\text{Ker}(\phi) := \{g \in G : \phi(g) = 1\}$. Show that $\text{Ker}(\phi)$ is a subgroup of $G$.

**Definition.** Given two groups $G$ and $H$, the product group is the group whose underlying set is $G \times H$, with multiplication structure $(g_1, h_1) * (g_2, h_2) = (g_1g_2, h_1h_2)$.

**Problem 3.5.** Let $G$ and $H$ be two groups and let $\pi_1$ and $\pi_2$ be the projections $G \times H \to G$ and $G \times H \to H$ onto the first and second factor. Show that $G \times H$ obeys the universal property of products, meaning that, for any group $F$ with maps $\phi_1 : F \to G$ and $\phi_2 : F \to H$, there is a unique map $(\phi_1, \phi_2) : F \to G \times H$ such that the diagram below commutes:

**Definition.** A group $G$ is called abelian if $g_1 * g_2 = g_2 * g_1$ for all $g_1, g_2 \in G$.

If $G$ is abelian, we will often denote $*$ by $+$ and 1 by 0. We will never use these notations for a non-abelian group.

**Problem 3.6.** Let $G$ be a group. Show that $G$ is abelian if and only if:

1. The map $g \mapsto g^{-1}$ is a group homomorphism.
2. The map $g \mapsto g^2$ is a group homomorphism.
3. The map $\mu : G \times G \to G$ by $\mu(g, h) = g * h$ is a group homomorphism.

We’ll toss in one more definition:

**Definition.** For $g \in G$, the conjugacy class of $g$ is the set $\text{Conj}(g) := \{hgh^{-1} : h \in G\}$.