Problem Set 4: Due Wednesday, February 5

Please see the course website for guidance on collaboration and formatting your problem sets.

Problem 4.1. Let \( 1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1 \) be a short exact sequence of groups. A right splitting of this sequence is a map \( \rho : C \to B \) such that \( \beta \circ \rho = \text{Id}_C \). Show that, if the sequence \( 1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1 \) has a right splitting, then \( B \cong A \times C \) for an action of \( A \) on \( C \). Hint: Apply Worksheet Problem 8.6 to the subgroups \( \alpha(A) \) and \( \rho(C) \) of \( B \).

Problem 4.2. Which of the following sequences are left split? Which are right split? (See Problems 3.6 and 4.1)

- (1) \( 1 \to C_2 \to C_0 \to C_3 \to 1 \).
- (2) \( 1 \to C_2 \to C_4 \to C_2 \to 1 \).
- (3) \( 1 \to A_5 \to S_5 \to \{\pm 1\} \to 1 \).
- (4) \( 1 \to \text{SL}_3(\mathbb{R}) \to \text{GL}_3(\mathbb{R}) \xrightarrow{\text{det}} \mathbb{R}^\times \to 1 \).

Problem 4.3. Let \( F \) be a field. Let \( \text{GL}_n(F) \) be the group of invertible \( n \times n \) matrices with entries in \( F \) and let \( \text{SL}_n(F) \) be the group of submatrices with determinant 1. The aim of this problem is to describe the abelianization of \( \text{GL}_n(F) \) and \( \text{SL}_n(F) \) in all cases.

For \( 1 \leq i \neq j \leq n \) and \( r \in F \), we define \( E_{ij}(r) \) to be the matrix with ones on the diagonal, an \( r \) in position \( (i, j) \) and zeroes everywhere else; we call such a matrix an elementary matrix. We showed in Math 593 (and you may use) that the elementary matrices generate \( \text{SL}_n(F) \).

1. Suppose that \( n \geq 3 \). Show that the elementary matrices are in the commutator subgroup of \( \text{SL}_n(F) \). Conclude that the abelianization of \( \text{SL}_n(F) \) is trivial and the abelianization of \( \text{GL}_n(F) \) is \( F^\times \). Hint: First think about matrices of the form \[
\begin{bmatrix}
0 & 1 & * \\
0 & 0 & 1
\end{bmatrix}.
\]
2. Show that the elementary matrices are in the commutator subgroup of \( \text{GL}_2(F) \) for \( \#(F) > 2 \) and in the commutator subgroup of \( \text{SL}_2(F) \) for \( \#(F) > 3 \). Conclude that the abelianization of \( \text{SL}_2(F) \) is trivial and the abelianization of \( \text{GL}_2(F) \) is \( F^\times \) in these cases. Hint: First think about matrices of the form \[
\begin{bmatrix}
0 & * \\
0 & 1
\end{bmatrix}.
\]
3. What are the abelianizations of \( \text{GL}_2(F_2) = \text{SL}_2(F_2) \) and \( \text{SL}_2(F_3) \)?

Problem 4.4. Let \( A \) be an abelian group and let \( \phi : A \to A \) be an automorphism. Let \( \mathbb{Z} \) act on \( A \) by \( k : a \mapsto \phi^k(a) \); by a standard abuse of notation, we will also denote this action by \( \phi \). Let \( G = A \rtimes_{\phi} \mathbb{Z} \). Show that the abelianization of \( G \) is isomorphic to \( A/([\text{Id} - \phi](A)) \times \mathbb{Z} \). In this formula, \( \text{Id} - \phi \) is an additive map \( A \to A \) and we are quotienting by its image.

The remaining problems are not tightly tied to the current material; but many of them will be useful in the future.

Problem 4.5. Let \( G \) be a finite group.

1. Let \( X \) be a finite set with a transitive action of \( G \), and \( |X| > 1 \). Show that there is some \( g \in G \) which fixes no element of \( X \). Hint: What lemma have we proved involving the number of fixed points?
2. Let \( H \subseteq G \) be a proper subgroup of \( G \). Show that there is some conjugacy class \( C \) of \( G \) with \( H \cap C = \emptyset \).

Problem 4.6. Let \( p \) be a prime number. For a positive integer \( n \), let \( v(n) \) be the integer such that \( p \) divides \( n! \) precisely \( v(n) \) times.

1. Write \( n = mp + r \) for \( 0 \leq r \leq p - 1 \). Show that \( v(n) = m + v(m) \).
2. Show that \( S_n \) contains a subgroup of order \( p^{v(n)} \). Hint: You might find it a good warm up to do the cases \( n = mp \) for \( m < p \) and \( n = p^2 \) first.

Problem 4.7. Let \( k \) be a finite field with \( q \) elements and let \( 1 \leq m \leq n \). Show that the number of \( m \times n \) matrices with entries in \( k \) and rank \( m \) is \( \prod_{j=0}^{m-1}(q^n - q^j) \). (Hint: Induct on \( m \).)

Problem 4.8. Let \( R \) be a ring (not assumed commutative). An element \( x \in R \) is called nilpotent if there is some positive integer \( m \) for which \( x^m = 0 \). A (two-sided) ideal whose every element is nilpotent is called nil.

1. Show that, if \( x \) is nilpotent, then \( 1 + x \) is a unit.
2. Let \( N \) be a nil ideal of \( R \). Let \( U = \{1 + x : x \in N\} \). Show that \( U \) is a group under multiplication.