Problem 5.1. Recall that \( Z(G) \) is the center of a group \( G \), and that a central series of \( G \) is a sequence of subgroups \( G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N \) where \( G_{i+1}/G_i \subseteq Z(G/G_i) \) for all \( i \).

1. The upper (or ascending) central series of \( G \) is defined inductively by \( U_0 = \{ e \} \) and \( U_{k+1} = \pi_k^{-1}(Z(G/U_k)) \), where \( \pi_k \) is the projection \( G \to G/U_k \). Let \( \{ e \} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \) be a central series with \( G_0 = \{ e \} \). Show that \( G_k \subseteq U_k \).

2. The lower (or descending) central series is defined inductively by \( L^0 = G \) and letting \( L^{k+1} \) be the group generated by all products \( ghg^{-1}h^{-1} \) with \( g \in G \) and \( h \in L^k \). Let \( G = G^0 \triangleright G^1 \triangleright G^2 \triangleright \cdots \) be a central series with \( G^0 = G \) (note that we have reversed the direction of the numbering). Show that \( L^k \subseteq G^k \).

3. Recall that a group \( G \) is called nilpotent if there is a central series \( \{ e \} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N = G \). Show that \( G \) is nilpotent, if and only if \( U_k \) is eventually \( G \), if and only if \( L^k \) is eventually \( \{ e \} \).

Problem 5.2. Let \( R \) be a ring (not assumed commutative) and let \( I \) be a two sided ideal of \( R \). We define \( I^m \) to be the two sided ideal generated by all products \( x_1 x_2 \cdots x_m \) for \( x_1, x_2, \ldots, x_m \in I \). We define the ideal \( N \) to be nilpotent if there is a positive integer \( m \) such that \( N^m = \{ 0 \} \). Let \( N \) be a nilpotent ideal and let \( U \) be the group \( \{ 1 + x : x \in N \} \). Show that \( U \) is a nilpotent group.\(^1\)

The next three problems are adapted from QR exams.

Problem 5.3. Let \( G \) be a group where \( g^2 h^2 = h^2 g^2 \) for all \( g \) and \( h \).

1. Let \( N \) be the subgroup \( \langle g^2 \mid g \in G \rangle \). Show that \( N \) is normal.

2. Show that \( G \) is solvable.

Problem 5.4. A group \( G \) is called virtually solvable if \( G \) has a normal subgroup \( N \) such that \( N \) is solvable and \( G/N \) is finite.

1. Show that a subgroup of a virtually solvable group is virtually solvable.

2. Show that a quotient of a virtually solvable group is virtually solvable.

Problem 5.5. Let \( G_1 \) and \( G_2 \) be groups and let \( S \) be a subgroup of \( G_1 \times G_2 \). Let \( H_i \) be the projection of \( S \) onto \( G_i \) and let \( K_i = S \cap G_i \).

1. Show that \( K_i \) is normal in \( H_i \).

2. Show that \( H_1/K_1 \cong H_2/K_2 \).

These problems do not use the current material, but will be useful in the future. Let \( k \) be a field and let \( k[x] \) be the ring of polynomials with coefficients in \( x \). Recall that \( k[x] \) is a PID; you may use this fact freely in these problems.

Problem 5.6. Let \( K \subset L \) be two fields and let \( a(x) \) and \( b(x) \in K[x] \). Let \( g(x) \) be the GCD of \( a \) and \( b \) in \( K[x] \). Show that \( g(x) \) is also the GCD of \( a \) and \( b \) in \( L[x] \).

Problem 5.7. For a polynomial \( f(x) = \sum f_j x^j \in k[x] \), we define the derivative \( f'(x) \) to be \( \sum j f_j x^{j-1} \).

1. For any two polynomials \( f(x) \) and \( g(x) \in k[x] \), show that \( (f + g)'(x) = f'(x) + g'(x) \) and \( (fg)'(x) = f(x)g'(x) + f'(x)g(x) \). Note that your proof should work for any field \( k \).

For a nonzero polynomial \( f(x) \) and an irreducible polynomial \( p(x) \), let \( m_p(f) \) be the number of times that \( p \) appears in the prime factorization of \( f \).

2. Let \( f \) and \( p \) be as above and suppose that \( m_p(f) > 0 \). Show that, if \( k \) has characteristic zero, then \( m_p(f') = m_p(f) - 1 \).

3. Give an example to show that the above need not be true in nonzero characteristic.

\(^1\)This is the most direct conceptual connection I know between the uses of the word “nilpotent” in ring theory and in group theory. The actual historical origins of the word come from the following related case: Start with \( R \) being \( n \times n \) upper-triangular real matrices, \( N \) being upper-triangular real matrices with 0’s on the diagonal and \( U \) being upper-triangular real matrices with 1’s on the diagonal. Let \( G \) be a Lie subgroup of \( U \), then the Lie algebra \( g \) is a Lie subalgebra of \( N \). The group \( G \) is a nilpotent group, and the Lie algebra \( g \) is a nilpotent Lie algebra. (We have not defined Lie groups, Lie algebras or what it means for a Lie algebra to be nilpotent.)