5. Normal Subgroups, Quotient Groups, Short Exact Sequences

Problem 5.1. Let $G$ be a group and let $N$ be a subgroup. Show that the following are equivalent:

1. For all $g \in G$, we have $gNg^{-1} = N$.
2. All elements of $G/N$ have the same stabilizer, for the left action of $G$ on $G/N$.
3. Every left coset of $N$ in $G$ is also a right coset.
4. If $g_1N = g_1'N$ and $g_2N = g_2'N$, then $g_1g_2N = g_1'g_2N$.

**Definition.** A subgroup $N$ obeying the equivalent conditions of Problem 5.1 is called a normal subgroup of $G$. We write $N \trianglelefteq G$ to indicate that $N$ is a normal subgroup of $G$.

Problem 5.2. Let $G$ be $S_3$. Which of the following subgroups are normal?

1. The subgroup generated by (12).
2. The subgroup generated by (123).

Problem 5.3. Let $G$ be a group and let $N$ be a normal subgroup of $G$.

1. Prove or disprove: Let $\alpha : F \to G$ be a group homomorphism. Then $\alpha^{-1}(N)$ is normal in $F$.
2. Prove of disprove: Let $\beta : G \to H$ be a group homomorphism. Then $\beta(N)$ is normal in $H$.
3. At least one of the statements above is false. Find an additional hypothesis you could add to make it true.

**Definition.** Given a group $G$ and an normal subgroup $N$, the quotient group $G/N$ is the group whose underlying set is the set of cosets $G/N$ with multiplication such that $(g_1N)(g_2N) = g_1g_2N$.

This definition makes sense by Part (4) of Problem 5.1. I won’t make you check that this is a group, but do so on your own time if you have any doubt. Also, I won’t make you check this, but the groups $G/N$ and $N \setminus G$, defined in the obvious ways, are isomorphic.

Let $\phi : G \to H$ be a group homomorphism. Recall that the image and kernel of $\phi$ are $\text{Ker}(\phi) := \{g \in G : \phi(g) = 1\}$ and $\text{Im}(\phi) := \{\phi(g) : g \in G\}$.

Problem 5.4. Show that the kernel of $\phi$ is a normal subgroup of $G$.

Problem 5.5. Show that the “obvious” map from $G/\text{Ker}(\phi)$ to $\text{Im}(\phi)$ is an isomorphism.

We often discuss quotients using the language of short exact sequences:

**Definition.** A short exact sequence $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ is three groups $A$, $B$ and $C$, and two group homomorphisms $\alpha : A \to B$ and $\beta : B \to C$ such that $\alpha$ is injective, $\beta$ is surjective, and $\text{Im}(\alpha) = \text{Ker}(\beta)$.

I will occasionally write 0 instead of 1 at one end or the other of a short exact sequence. I do this when the adjacent group (meaning $A$ or $C$) is abelian and it would feel bizarre to denote the identity of that abelian group as 1.

We’ll write $C_n$ for the abelian group $\mathbb{Z}/n\mathbb{Z}$. This is called the cyclic group of order $n$.

Problem 5.6. Show that there is a short exact sequence $1 \to C_m \to C_{mn} \to C_n \to 1$.

Problem 5.7. Show that there is a short exact sequence $1 \to C_3 \to S_3 \to S_2 \to 1$.

Problem 5.8. Show that there is a short exact sequence $1 \to C_2^2 \to S_4 \to S_3 \to 1$.

Problem 5.9. What is the relationship between Problems 5.7 and 5.8 and your computations on the first day of class involving $\{(\beta_1 + \omega\beta_2 + \omega^2\beta_3)^3, (\beta_1 + \omega^2\beta_2 + \omega\beta_3)^3\}$ and $\{(\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4)^2, (\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4)^2, (\gamma_1 - \gamma_2 - \gamma_3 + \gamma_4)^2\}$?