5. Group actions

**Definition:** Let $G$ be a group and let $X$ be a set. An action of $G$ on $X$ is a map $*: G \times X \rightarrow X$ obeying $(g_1 * g_2) \ast x = g_1 \ast (g_2 \ast x)$ and $e \ast x = x$.

Depending on context, we may denote $*$ by $\ast$, $\times$, $\cdot$ or no symbol at all. This notion of an action can also be called a “left action”; a “right action” is a map $*: X \times G \rightarrow X$ obeying $x \ast (g_2 \ast g_1) = (x \ast g_2) \ast g_1$.

**Problem 5.1.** Let $G \times X \rightarrow X$ be a left action of $G$ on $X$. Define a map $X \times G \rightarrow X$ by $(x, g) \mapsto g^{-1} \ast x$. Show that this is a right action of $G$ on $X$.

**Problem 5.2.** Let $S_X$ be the group of bijections $X \rightarrow X$, with the group operation of composition. Show that an action of $G$ on $X$ is the same as a group homomorphism $G \rightarrow S_X$.

**Definition:** Let $G$ be a group which acts on a set $X$. For $x \in X$, the stabilizer $\text{Stab}(x)$ of $x$ is \{ $g \in G : g \ast x = x$ \}. For $g \in G$, the fixed points $\text{Fix}(g)$ of $g$ are \{ $x \in X : g \ast x = x$ \}.

**Problem 5.3.** With $G$, $X$ and $x$ as above, show that $\text{Stab}(x)$ is a subgroup of $X$.

**Problem 5.4.** Let $G$, $X$ and $x$ be as above and let $g \in G$. Show that $\text{Stab}(gx) = g \ast \text{Stab}(x)g^{-1}$.

**Definition:** For $G$, $X$ and $x$ as above, the orbit of $x$, written $Gx$, is \{ $gx : g \in G$ \}.

**Problem 5.5.** (*The Orbit-Stabilizer theorem*) If $G$ is finite, show that $\#(G) = \#(Gx) \#(\text{Stab}(x))$.

The set of orbits of $G$ on $X$ is denoted $G \backslash X$. If we have a right action, we write $X/G$.

**Problem 5.6.** (*Burnside’s Lemma*) Let $G$ be a finite group and let $X$ be a finite set on which $G$ acts. Show that

$$
\frac{1}{\#G} \sum_{g \in G} \#\text{Fix}(g) = \#(G \backslash X).
$$

**Definition:** Let $G$ be a group and let $H$ be a subgroup. Let $H$ act on $G$ by $h \ast g = hg$. The orbits of this action are called the right cosets of $H$ in $G$. The left cosets are the orbits for the right action $G \ast H \rightarrow G$. The number of cosets of $H$ in $G$ is called the index of $H$ in $G$ and written $[G : H]$.

**Problem 5.7.** Show that $G$ has a left action on the set $G/H$ of left cosets, such that $g_1 \ast (g_2 H) = (g_1 \ast g_2)H$. Show that the stabilizer of the coset $eH$ is $H$.

**Problem 5.8.** (*Lagrange’s Theorem*) Let $G$ be a finite group and let $H$ be a subgroup. Show that $\#(H)$ divides $\#(G)$.

**Problem 5.9.** Let $G$ be a finite group with $\#(G) = N$. Let $g \in G$ and let the group generated by $g$ have $n$ elements.

1. Show that $n$ divides $N$.
2. Show that $g^N = 1$.

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