5. Group actions

**Definition:** Let $G$ be a group and let $X$ be a set. An action of $G$ on $X$ is a map $*: G \times X \to X$ obeying $(g_1 * g_2) * x = g_1 * (g_2 * x)$ and $e * x = x$.

Depending on context, we may denote $*$ by $\cdot$, $\times$, or no symbol at all. This notion of an action can also be called a “left action”; a “right action” is a map $*: X \times G \to X$ obeying $x * (g_2 * g_1) = (x * g_2) * g_1$.

**Problem 5.1.** Let $G \times X \to X$ be a left action of $G$ on $X$. Define a map $X \times G \to X$ by $(x, g) \mapsto g^{-1}x$. Show that this is a right action of $G$ on $X$.

**Problem 5.2.** Let $G$ act on a set $X$ by the orbit-stabilizer theorem. For $x \in X$, the equilibrium of $x$ is $\{g \in G : g x = x\}$. Show that this is a right action of $G$ on $X$.

**Problem 5.3.** With $G$, $X$ and $x$ as above, show that $\text{Stab}(x)$ is a subgroup of $X$.

**Problem 5.4.** Let $G$, $X$ and $x$ be as above and let $g \in G$. Show that $\text{Stab}(g x) = g \text{Stab}(x) g^{-1}$.

**Definition:** For $G$, $X$ and $x$ as above, the orbit of $x$, written $Gx$, is $\{g x : g \in G\}$.

**Problem 5.5.** (The Orbit-Stabilizer theorem) If $G$ is finite, show that $\#(G) = \#(Gx) \#(\text{Stab}(x))$.

The set of orbits of $G$ on $X$ is denoted $G \backslash X$. If we have a right action, we write $X / G$.

**Problem 5.6.** (Burnside’s Lemma) Let $G$ be a finite group and let $X$ be a finite set on $G$ acts. Show that $\frac{1}{\#G} \sum_{g \in G} \#\text{Fix}(g) = \#(G \backslash X)$.

**Definition:** Let $G$ be a group and let $H$ be a subgroup. Let $H$ act on $G$ by $h * g = hg$. The orbits of this action are called the right cosets of $H$ in $G$. The left cosets are the orbits for the right action $G * H \to G$. The number of cosets of $H$ in $G$ is called the index of $H$ in $G$ and written $[G : H]$.

**Problem 5.7.** Show that $G$ has a left action on the set $G / H$ of left cosets, such that $g_1 * (g_2 H) = (g_1 * g_2) H$. Show that the stabilizer of the coset $e H$ is $H$.

**Problem 5.8.** (Lagrange’s Theorem) Let $G$ be a finite group and let $H$ be a subgroup. Show that $\#(H)$ divides $\#(G)$.

**Problem 5.9.** Let $G$ be a finite group with $\#(G) = N$. Let $g \in G$ and let the group generated by $g$ have $n$ elements.

1. Show that $n$ divides $N$.
2. Show that $g^n = 1$.

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