Problem 7.1. Let $K$ and $L$ be fields and let $\phi : K \to L$ be a ring homomorphism (this includes the condition that $\phi(1) = 1$). Show that $\phi$ is injective.

Problem 7.2. Let $A$ be a finite abelian group such that, for every integer $d$, $A$ contains at most $d$ solutions to $a^d = 1$. Show that $A$ is cyclic. You may use the classification of finite abelian groups.

Problem 7.3. Let $n$ be a positive integer. Let $L$ be a field in which $n \neq 0$ and where the polynomial $x^n - 1$ factors into linear factors $x^n - 1 = \prod(x - \theta_j)$. Then $L$ contains $n$ roots of unity.

1. Show that the $\theta_j$ are distinct.
2. Show that the $\theta_j$ form a cyclic subgroup of $L^\times$. (Hint: The most efficient route is problem 7.2.) A generator of this group is called a primitive $n$-th root of unity in $L$.
3. Let $\sigma$ be an automorphism of $L$. Show that there is some integer $a$, relatively prime to $n$, such that $\sigma(\theta) = \theta^a$ for each root $\theta$ of $x^n - 1$.

Problem 7.4. Let $K$ be a field of characteristic $p$. Let $a$ be an element of $K$ such that the polynomial $x^p - x - a$ is irreducible over $k$, and let $L = K[t]/(t^p - t - a)K[t]$.

1. Show that there are $b$ and $c \in K$ such that $L \cong K[x]/(x^2 + bx + c)K[x]$.
2. Suppose that $K$ does not have characteristic 2. Show that $L \cong K[\sqrt{b^2 - 4c}]$.

Problem 7.5. Let $L/K$ be a field extension of degree 2 (meaning that $L$ has dimension 2 as a $K$-vector space).

1. Show that there are $b$ and $c \in K$ such that $L \cong K[x]/(x^2 + bx + c)K[x]$.
2. Suppose that $K$ does not have characteristic 2. Show that $L \cong K[\sqrt{b^2 - 4c}]$.

Problem 7.6. (Reduced Row Echelon form) Let $M$ be a $d \times n$ matrix with entries in a field $k$. Let $1 \leq j_1 < j_2 < \cdots < j_d \leq n$. We will say that $M$ is in reduced row echelon form with pivots in columns $j_1, j_2, \ldots, j_d$ if

$$M_{rj} = \begin{cases} 1 & j = j_r \\ 0 & j = j_s \text{ for } s \neq r \\ 0 & j < j_r \end{cases}$$

We say that $M$ is in reduced row echelon form if it is in reduced row echelon for some set of columns $j_1, j_2, \ldots, j_d$. The aim of this problem is to check that every $d$-dimensional subspace of $k^n$ is the row span of exactly one $d \times n$ matrix in reduced row echelon form.

1. Let $M$ be in reduced row echelon form with pivots in columns $j_1, j_2, \ldots, j_d$ and let $V \subseteq k^n$ be its row span. Show how to recover the $j_1, j_2, \ldots, j_d$ from the data of how $V$ intersects various subspaces of $k^n$.
2. Let $V$ be a $d$-dimensional subspace of $k^n$. Show that there is at most one matrix $M$ in reduced row echelon form with row span $V$.
3. Let $V$ be a $d$-dimensional subspace of $k^n$. Show that there is at least one matrix $M$ in reduced row echelon form with row span $V$.

Problem 7.7. Let $L$ be a field, let $H$ be a group of automorphisms of $L$ and let $F$ be the elements of $L$ fixed by $H$. Suppose that $V$ is an $L$-vector subspace of $L^n$ and that $H$ takes $V$ to itself. Show that $V$ has a basis whose elements lie in $F^n$. (Hint: Look at Problem 7.4.)

Problem 7.8. Let $K$ be a field of characteristic $p$. For $\theta \in K$, define $F(\theta) = \theta^p$ (this is called the Frobenius map).

1. Show that $F : K \to K$ is a field homomorphism.
2. Show that $F$ is injective.
3. If $K$ is finite, show that $F$ is bijective.