Problem 8.1. Let \( n \) be a positive integer and let \( K \) be a field in which \( n \neq 0 \). Let \( c \) be a nonzero element of \( K \) and let \( L \) be a splitting field of \( x^n - c \).

1. Show that the polynomial \( x^n - 1 \) splits in \( L \).

Let \( \zeta \) generate the cyclic group of roots of \( x^n - 1 \) in \( L \). Let \( \gamma \) denote one of the roots of \( x^n - c \) in \( L \).

2. Show that \( K(\zeta) \) is a splitting field for \( x^n - 1 \) and \( \text{Aut}(K(\zeta)/K) \) is isomorphic to a subgroup of the unit group \((\mathbb{Z}/n\mathbb{Z})^\times\).

3. Let \( \sigma \in \text{Aut}(L/K) \). Show that there are integers \( a \in (\mathbb{Z}/n\mathbb{Z})^\times \) and \( b \in \mathbb{Z}/n\mathbb{Z} \) such that \( \sigma(\zeta) = \zeta^a \) and \( \sigma(\gamma) = \zeta^b\gamma \).

4. Show that \( \text{Aut}(L/K) \) is isomorphic to a subgroup of \((\mathbb{Z}/n\mathbb{Z})^\times \ltimes \mathbb{Z}/n\mathbb{Z}\).

Problem 8.2. Let \( K \) have characteristic not 2. Let \( f(x) = x^4 + bx^2 + c \) and let \( L \) be a splitting field of \( f \). We assume that \( f \) is separable, and we number the roots of \( f \) so that \( \theta_3 = -\theta_1 \) and \( \theta_4 = -\theta_2 \).

1. Show that \( \text{Aut}(L/K) \) is contained in the group of symmetries of the square shown below:

\[
\begin{array}{c|c|c}
1 & 2 & \hline \\
4 & \hline & 3
\end{array}
\]

2. Check that \((\theta_1\theta_2)^2 = c \) and \((\theta_1^2 - \theta_2^2)^2 = b^2 - 4c\).

3. Show that \( \text{Aut}(L/K) \) in contained in the subgroup \((13), (24)\) of \( S_4 \) if and only if \( b^2 - 4c \) is square in \( K \).

4. Show that \( \text{Aut}(L/K) \) in contained in the subgroup \((12)(34), (14)(23)\) of \( S_4 \) if and only if \( c \) is square in \( K \).

5. Show that \( \text{Aut}(L/K) \) in contained in the subgroup \((1234)\) of \( S_4 \) if and only if \( c(b^2 - 4c) \) is square in \( K \).

Problem 8.3. Let \( L/K \) be a field extension of degree \( n \). For \( \theta \in L \), let \( m_\theta \) be the map \( x \mapsto \theta x \) from \( L \) to \( L \). Let \( f(x) = x^d + f_{d-1}x^{d-1} + \cdots + f_0 \) be the minimal polynomial of \( \theta \) over \( K \).

1. Show that \( m_\theta \) is a \( K \)-linear map.

2. Express the minimal polynomial and characteristic polynomial of \( m_\theta \) in terms of \( f(x) \), \( d \) and \( n \).

3. The trace \( T_{L/K}(\theta) \) is defined to be the trace of the linear map \( m_\theta \). Express \( T_{L/K}(\theta) \) in terms of the \( f_j \), \( d \) and \( n \).

4. The norm \( N_{L/K}(\theta) \) is defined to be the determinant of the linear map \( m_\theta \). Express \( N_{L/K}(\theta) \) in terms of the \( f_j \), \( d \) and \( n \).

Problem 8.4. Let \( p \) be a prime and let \( q = p^n \).

1. Let \( k \) be a field of characteristic \( p \). Show that the roots of \( x^q = x \) in \( k \) form a subfield of \( k \).

2. Define \( \mathbb{F}_q \) to be the splitting field of \( x^q - x \) over \( \mathbb{Z}/p\mathbb{Z} \). Show that \( \mathbb{F}_q \) is a field with \( q \) elements.

3. Let \( F \) be any field with \( q \) elements. Show that \( F \) is a splitting field for \( x^q - x \) over \( \mathbb{F}_p \), and conclude that \( F \cong \mathbb{F}_q \).