**Definition:** A subnormal series of a group $G$ is a chain of subgroups $G_0 < G_1 < G_2 < G_3 < \cdots < G_N \subseteq G$ where $G_{j-1}$ is normal in $G_j$. A composition series is a subnormal series where $G_0 = \{e\}$, $G_N = G$ and each subquotient $G_j/G_{j-1}$ is simple. A quasi-composition series is a composition series where $G_0 = \{e\}$, $G_N = G$ and each subquotient is either simple or trivial.

**Problem 8.1.** Show that a group which has a quasi-composition series has a composition series.

**Problem 8.2.** Show that every finite group has a composition series.

**Problem 8.3.** Show that $S_4$ has a composition series with subquotients $C_2$, $C_2$, $C_3$ and $C_2$.

**Problem 8.4.** Show that $\text{GL}_2(\mathbb{F}_7)$ has a composition series with subquotients $C_2$, $\text{PSL}_2(\mathbb{F}_7)$, $C_2$ and $C_3$. You may assume that $\text{PSL}_2(\mathbb{F}_7)$ is simple. (For a field of characteristic $\neq 2$, the group $\text{PSL}_2(F)$ is $\text{SL}_2(F)/\pm \text{Id}$. See the worksheet on simple groups for the definition of $\text{PSL}_n(F)$ in general.)

**Problem 8.5.** Let $1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$ be a short exact sequence, and let $\{1\} = A_0 \subset A_1 \subset \cdots \subset A_a = A$ and $\{1\} = C_0 \subset C_1 \subset \cdots \subset C_c = C$ be composition series of $A$ and $C$. Show that
\[ \{1\} = \alpha(A_0) \subset \alpha(A_1) \subset \cdots \alpha(A_a) = \beta^{-1}(C_0) \subset \beta^{-1}(C_1) \subset \cdots \subset \beta^{-1}(C_c) = B \]
is a composition series for $B$.

**Problem 8.6.** Let $1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$ be a short exact sequence and let $\{1\} = B_0 \subset B_1 \subset \cdots \subset B_b = B$ be a composition series of $B$.

1. Show that $\{1\} = \alpha^{-1}(B_0) \subset \alpha^{-1}(B_1) \subset \cdots \subset \alpha^{-1}(B_b) = A$ is a quasi-composition series for $A$.
2. Show that $\{1\} = \beta(B_0) \subset \beta(B_1) \subset \cdots \subset \beta(B_b) = C$ is a quasi-composition series for $C$.

We are setting up to prove the Jordan-Holder theorem for groups. Here is a useful lemma.

**Problem 8.7.** Let $1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$ be a short exact sequence and let $B'$ be a normal subgroup of $B$. Set $A' = \alpha^{-1}(B)$ and $C' = \beta(B)$. You might find it useful to think of $A$ as a subgroup of $B$, and $A'$ as $A \cap B'$.

1. Show that $1 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 1$ is a short exact sequence.
2. Show that $1 \rightarrow A/A' \rightarrow B/B' \rightarrow C/C' \rightarrow 1$ is a short exact sequence.