9. The Jordan-Holder Theorem

We recall the definitions from last time:

**Definition:** A subnormal series of a group G is a chain of subgroups $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_N = \{ e \}$ where $G_{j+1}$ is normal in $G_j$. A composition series is a subnormal series where each subquotient $G_j / G_{j+1}$ is simple. A generalized composition series is a composition series where each quotient is either simple or trivial.

Let $G$ be a group with a composition series $\{ e \} = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_N = G$. We define $N$ to be the length of the composition series and write $N = \ell(G)$. For a simple group $\Gamma$ and a composition series $G_\bullet$, we define $m(G_\bullet, \Gamma)$ to be the number of quotients $G_j / G_{j+1}$ which are isomorphic to $\Gamma$. Our aim today is to prove

**Theorem (Jordan-Holder):** Let $G$ be a group and let $G_\bullet$ and $G'_\bullet$ be two composition series for $G$. Then $\ell(G_\bullet) = \ell(G'_\bullet)$ and, for any simple group $\Gamma$, we have $m(G_\bullet, \Gamma) = m(G'_\bullet, \Gamma)$.

Let $1 \to A \mathrel{\to} B \mathrel{\to} C \to 1$ be a short exact sequence of groups. Let $B_\bullet$ be a composition series for $B$. Recall that we proved on the previous worksheet that $\{ 1 \} = \alpha^{-1}(B_0) \subseteq \alpha^{-1}(B_1) \subseteq \cdots \subseteq \alpha^{-1}(B_b) = A$ is a quasi-composition series for $A$ and $\{ 1 \} = \beta(B_0) \subseteq \beta(B_1) \subseteq \cdots \subseteq \beta(B_b) = C$ is a quasi-composition series for $C$.

**Problem 9.1.** With the above notations, let $A_\bullet$ and $C_\bullet$ be the composition series obtained from deleting duplicate entries from the quasi-composition series above.

1. Show that $\ell(B_\bullet) = \ell(A_\bullet) + \ell(C_\bullet)$.
2. For any simple group $\Gamma$, show that $m(B_\bullet, \Gamma) = m(A_\bullet, \Gamma) + m(C_\bullet, \Gamma)$.

At this point, you have enough to prove the Jordan-Holder theorem for finite groups, by induction on $\#(G)$.

**Problem 9.2.** Check the base case: Jordan-Holder holds for the trivial group.

**Problem 9.3.** Check also that Jordan-Holder holds for simple groups.

**Problem 9.4.** Suppose that $G$ is a finite group which is neither simple nor trivial, and suppose that Jordan-Holder holds for all groups of size less than $\#(G)$. Show that Jordan-Holder holds for $G$. This completes the induction, for $\#(G) < \infty$.

The Jordan-Holder theorem is also true for infinite groups that have composition series! Proving this requires no big new ideas, but a little more finesse. Define $L(G) = \min \ell(G_\bullet)$, where the minimum is over all composition series for $G$. Note $L(G) = 0$ if and only if $G$ is trivial, and $L(G) > 0$ for any nontrivial $G$.

**Problem 9.5.** Check that $L(G) = 1$ if and only if $G$ is simple.

**Problem 9.6.** Let $1 \to A \to B \to C \to 1$ be a short exact sequence of groups.

1. Show that $L(B) \geq L(A) + L(C)$.
2. If $A$ and $C$ are nontrivial, show that $L(B) > L(A)$ and $L(B) > L(C)$.

**Problem 9.7.** Prove the Jordan-Holder theorem by induction on $L(G)$.

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1 In fact, equality holds and you have the tools to show it, but you don’t need this.