A. CENTER, CENTRAL SERIES AND NILPOTENT GROUPS

A particularly nice sort of subnormal series is a central series, and a particularly nice kind of solvable group is a nilpotent group.

**Definition:** The center of a group $G$ is the set $Z(G) := \{ h : gh = hg \forall g \in G \}$.

**Problem A.1.** Let $G$ be a group.

1. Check that $Z(G)$ is a normal subgroup of $G$.
2. Check that every subgroup of $Z(G)$ is normal in $G$.

**Problem A.2.** Let $k$ be a field and let $U$ be the group of matrices with entries in $k$ of the form $\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$. Show that the center of $U$ is the group of matrices of the form $\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$.

This problem was on the problem sets; check that everyone in your group remembers how to do it.

**Problem A.3.** Let $p$ be a prime and let $G$ be a group of order $p^k$ for $k \geq 1$. Show that $Z(G)$ is nontrivial.

**Definition:** Let $G$ be a group. A central series of $G$ is a sequence of subgroups $G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N$ such that, if $g \in G$ and $h \in G_i$ then $ghg^{-1}h^{-1} \in G_{i-1}$, for $1 \leq i \leq N$. $G$ is called nilpotent if it has a central series $G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N$ with $G_0 = \{e\}$ and $G_N = G$.

**Problem A.4.** Let $G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N$ be a series of subgroups of $G$. Show that $G$ is a central series if and only if all the $G_i$ are normal in $G$, and $G_i/G_{i-1} \subseteq Z(G/G_{i-1})$ for $1 \leq i \leq N$.

**Problem A.5.** Let $k$ be a field and let $U$ be the group of matrices with entries in $k$ of the form

$$
\begin{bmatrix}
1 & * & \cdots & * \\
1 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
1 & \cdots & * & 1
\end{bmatrix}
$$

Show that $U$ is nilpotent.

**Problem A.6.** Let $p$ be a prime and let $G$ be a group of order $p^k$ for some $k \geq 1$. Show that $G$ is nilpotent.

There is a converse to Problem A.6 which I hope to prove: The finite nilpotent groups are precisely the direct products of $p$-groups.

**Problem A.7.** Show that a nilpotent group is solvable.

**Problem A.8.** Show that a subgroup of a nilpotent group is nilpotent.

**Problem A.9.** Show that a quotient of a nilpotent group is nilpotent.