B. Finite nilpotent groups are products of \( p \)-groups.

Today’s goal is to show:

**Theorem:** Let \( G \) be a finite group. Then \( G \) is nilpotent if and only if it is a direct product of \( p \)-groups.

**Problem B.1.** Show the easy direction: A direct product of \( p \)-groups is nilpotent.

From now on, let \( G \) be a finite nilpotent group with \( \#(G) = \prod p_i^{k_i} \). We will be proving, by induction on \( \#(G) \), that \( G \) is the direct product of its Sylow subgroups.

**Problem B.2.** Show that \( G \) has a central subgroup \( Z \) which is cyclic of prime order.

Let \( G' = G/Z \), so we have a short exact sequence \( 1 \to Z \to G \to G' \to 1 \). Let \( P_i' \) be a \( p_i \)-Sylow of \( G' \).

**By induction,** \( G' = \prod_i P_i' \). We number the prime factors of \( \#(G) \) such that \( \#(Z) = p_1 \). We analyze the Sylows of \( G \), starting with the \( p_1 \)-Sylow, and then the others.

**Problem B.3.**
1. Show that \( \beta^{-1}(P_1') \) is normal in \( G \).
2. Show that \( \beta^{-1}(P_1') \) is a \( p_{p_1} \)-Sylow of \( G \).

**Problem B.4.** Now, let \( i > 1 \). We have a short exact sequence \( 1 \to Z \to \beta^{-1}(P_i') \to P_i' \to 1 \).

1. Show that \( \beta^{-1}(P_i') \) is normal in \( G \).
2. Show that the \( p_i \)-Sylow of \( \beta^{-1}(P_i') \) is also a \( p_i \)-Sylow of \( G \).
3. Show that \( \beta^{-1}(P_i') \cong Z \times P_i' \) (here is where you use Schur-Zassenhaus).
4. Show that the \( p_i \)-Sylow of \( \beta^{-1}(P_i') \) is a characteristic subgroup of \( \beta^{-1}(P_i') \).
5. Show that the \( p_i \)-Sylow of \( G \) is normal in \( G \).

We have now shown that every Sylow subgroup of \( G \) is normal in \( G \).

**Problem B.5.** Conclude by proving that \( G \) is the direct product of its Sylow subgroups.